

# Supersymmetry for Systems with Unitary Disorder: Circular Ensembles

Martin R. Zirnbauer\*

*Institute for Theoretical Physics, UCSB, Santa Barbara, U.S.A.*

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A generalized Hubbard-Stratonovitch transformation relating an integral over random unitary  $N \times N$  matrices to an integral over Efetov's unitary  $\sigma$ -model manifold, is introduced. This transformation adapts the supersymmetry method to disordered and chaotic systems that are modeled not by a Hamiltonian but by their scattering matrix or time-evolution operator. In contrast to the standard method, no saddle-point approximation is made, and no massive modes have to be eliminated. This first paper on the subject applies the generalized Hubbard-Stratonovitch transformation to Dyson's Circular Unitary Ensemble. It is shown how to use a supersymmetric variant of the Harish-Chandra-Itzykson-Zuber formula to compute, in the large- $N$  limit, the  $n$ -level correlation function for any  $n$ . Nontrivial applications to random network models, quantum chaotic maps, and lattice gauge theory, are expected.

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\* Permanent address: Institut für Theoretische Physik, Universität zu Köln, Germany

## I. INTRODUCTION

The long wave length physics of disordered metals is governed by the so-called diffuson and cooperon modes describing the coherent long-range propagation of particles and holes. An efficient way to handle the weak-disorder perturbation expansion involving these modes is provided by the nonlinear  $\sigma$ -model representation pioneered by Wegner [1] and turned into a nonperturbative tool by Efetov [2]. The great advantage of Efetov's supersymmetric version of the  $\sigma$  model is that it applies not only to diffusive systems but also to systems in the ergodic and quantum limits and in the localized regime. In an exciting recent development, the nonlinear  $\sigma$  model was extended also to ballistic systems [3] and to deterministic chaotic systems [4].

As it stands, Efetov's method is applicable to disordered systems that are modeled by a Hamiltonian and where the disorder in the Hamiltonian is weak and close to Gaussian. It was not known until very recently how to extend it to a closely related class of models defined by a *unitary* operator rather than the Hamiltonian. One prominent example are random network models for, say, the random flux problem or the percolation transition in integer quantum Hall systems [5]. These are formulated in terms of random scattering matrices connecting incoming to outgoing channels at every node of a network. While there has been some progress in understanding the Hamiltonian (or anisotropic) limit of these models, the original *isotropic* models have so far defied analytical treatment. A second example of importance are periodically driven systems such as the quantum kicked rotor [6] or the quantum kicked top [7]. Some of these systems have been found to possess quasi-energy level statistics of a universal type given by Dyson's circular random-matrix ensembles. Others exhibit the quantum phenomenon of dynamical localization, which corresponds to the Anderson localization of disordered metallic wires, but a rigorous derivation of the correspondence is as yet lacking.

In the present paper I am going to introduce a novel technique, akin to the supersymmetry method of Efetov but different from it in its details, which will allow analytical progress to be made with all of the above models. The heart of the formalism is a kind of generalized Hubbard-Stratonovitch transformation that takes an integral over random unitary  $N \times N$  matrices (of the type that appears in the Gaussian integral representation of a two-particle Green's function for noninteracting electrons) and trades it directly for an integral over Efetov's  $\sigma$ -model space of unitary symmetry. The transformation is exact for all  $N$  including  $N = 1$ , no saddle-point approximation is made, and no massive modes need to be eliminated. The present paper is fairly technical as it aims to give a detailed exposition of the method along with the necessary proofs. The method is illustrated at the simple example of Dyson's Circular Unitary Ensemble (CUE). Other, less trivial applications that have already been worked out and are about to be published [8,9], treat the quantum kicked rotor and the network model of Chalker and Coddington.

The paper is organized as follows. In Sec. II A the generalized Hubbard-Stratonovitch transformation is introduced in the context of the calculation of the generating function for spectral correlations of the CUE. An integral over  $U(N)$  is replaced by an integral over a pair of supermatrices  $Z, \bar{Z}$ . The geometric meaning of these supermatrices is elucidated in Sec. II B, where it is argued that they are coset representatives for Efetov's coset space  $G/H$  of unitary symmetry. In the Appendix it is shown that the matrix elements of  $Z$  can be viewed as the parameters of a

generalized coherent state obtained by acting with Efetov's symmetry group  $G$  (or, rather, its complexification) on a certain lowest-weight module. By embedding this module into a Fock space of bosons and fermions we succeed in giving an easy proof of the generalized Hubbard-Stratonovitch transformation. In Sec. IIC the  $n$ -level correlation function of the CUE is calculated for all  $n$ , by using a supersymmetric version of the Harish-Chandra-Itzykson-Zuber formula. Dyson's Circular Orthogonal Ensemble and the Circular Ensemble of type  $C$  are discussed in Secs. III and IV, demonstrating the versatility of the new method. A summary and outlook is given in Sec. V.

The present work was inspired by unpublished results of Alexander Altland, who had developed a similar (but less useful) Hubbard-Stratonovitch scheme. I am grateful to him for many discussions of this subject.

## II. CIRCULAR UNITARY ENSEMBLE (TYPE A)

Consider  $U(N)$ , the unitary group in  $N$  dimensions. This group, as any unimodular Lie group, comes with a natural integration measure, the so-called Haar measure  $dU$ , which is invariant under left and right translations  $U \mapsto U_L U U_R$ . We normalize  $dU$  by  $\int_{U(N)} dU = 1$ .

The set of random unitary  $N \times N$  matrices  $U$  with probability distribution given by the Haar measure  $dU$ , is called Dyson's [10] Circular Unitary Ensemble (CUE) in  $N$  dimensions. The CUE plays a central role in the random-matrix modeling of chaotic and/or disordered systems with broken time-reversal symmetry. A new analytical method for treating the CUE is introduced below. For simplicity, the method will be illustrated at the example of the generating function

$$F_{n_+, n_-}(\{\theta, \varphi\}) = \int_{U(N)} dU \prod_{\alpha=1}^{n_+} \frac{\text{Det}(1 - e^{i\varphi_{+\alpha}} U)}{\text{Det}(1 - e^{i\theta_{+\alpha}} U)} \prod_{\beta=1}^{n_-} \frac{\text{Det}(1 - e^{-i\varphi_{-\beta}} U^\dagger)}{\text{Det}(1 - e^{-i\theta_{-\beta}} U^\dagger)}, \quad (1)$$

from which all information about the spectral correlations of the CUE can be extracted.  $\varphi_{+\alpha}$ ,  $\varphi_{-\beta}$  and  $\theta_{+\alpha}$ ,  $\theta_{-\beta}$  are angular variables. (The latter are given a small imaginary part to prevent the determinants in the denominator from becoming singular.) The method does *not* exploit the invariance of  $F_{n_+, n_-}$  under conjugation  $U \mapsto g U g^{-1}$ , nor does it make any reference to the eigenbasis of  $U$ . For this reason it can be easily extended to the calculation of more general types of correlation function, such as those relating to wave functions and transport coefficients.

### A. Generalized Hubbard-Stratonovitch Transformation

The method we will use is a variant of Efetov's supersymmetry technique [2], which has become a standard tool for treating Hamiltonian systems with Gaussian disorder. Efetov's method starts by the usual trick of expressing the ratio of determinants as a Gaussian superintegral:

$$\begin{aligned} F_{n_+, n_-}(\{\theta, \varphi\}) &= \int_{U(N)} dU \int D(\phi, \bar{\phi}; \chi, \bar{\chi}) \exp \left( -\bar{\phi}_{+\alpha}^k (\delta^{kl} - e^{i\theta_{+\alpha}} U^{kl}) \phi_{+\alpha}^l - \bar{\chi}_{+\alpha}^k (\delta^{kl} - e^{i\varphi_{+\alpha}} U^{kl}) \chi_{+\alpha}^l \right. \\ &\quad \left. - \bar{\phi}_{-\beta}^k (\delta^{kl} - e^{-i\theta_{-\beta}} \bar{U}^{lk}) \phi_{-\beta}^l - \bar{\chi}_{-\beta}^k (\delta^{kl} - e^{-i\varphi_{-\beta}} \bar{U}^{lk}) \chi_{-\beta}^l \right), \end{aligned}$$

where  $\phi_{+\alpha}^k$ ,  $\bar{\phi}_{+\alpha}^k$ ,  $\phi_{-\beta}^l$ , and  $\bar{\phi}_{-\beta}^l$  ( $k, l = 1, \dots, N$ ;  $\alpha = 1, \dots, n_+$ ;  $\beta = 1, \dots, n_-$ ) are complex commuting (or bosonic) variables, with the bar denoting complex conjugation, and  $\chi, \bar{\chi}$  are anticommuting (or fermionic) variables.  $U^{kl}$  are the matrix elements of the unitary  $N \times N$  matrix  $U$ . The symbol  $D(\phi, \bar{\phi}; \chi, \bar{\chi})$  denotes the "flat" Berezin integration measure, i.e. it is given by the product of the differentials of the commuting variables times the product of the partial derivatives with respect to the anticommuting variables. The summation convention is used throughout this paper. The integral converges for

$$\text{Im}\theta_{+\alpha} > 0 > \text{Im}\theta_{-\beta}.$$

To simplify the notation we introduce a composite index  $a \equiv (\alpha, \sigma)$  where  $\sigma$  takes values  $\sigma = \text{B}$  (Bosons) or  $\sigma = \text{F}$  (Fermions). We then define a tensor  $\psi$  with  $(n_- + n_+)N =: nN$  components  $\psi_{\pm a}^i$  by

$$\psi_{\pm(\alpha, \text{B})}^i = \phi_{\pm\alpha}^i, \quad \psi_{\pm(\alpha, \text{F})}^i = \chi_{\pm\alpha}^i.$$

$\bar{\psi}$  is defined similarly. Setting  $D(\phi, \bar{\phi}; \chi, \bar{\chi}) \equiv D(\psi, \bar{\psi})$  and combining the phases  $\{\theta, \varphi\}$  into a single symbol  $\{\omega\}$  by

$$\omega_{\pm(\alpha,B)} = \theta_{\pm\alpha}, \quad \omega_{\pm(\alpha,F)} = \varphi_{\pm\alpha},$$

we may express the generating function in the abbreviated form

$$F_{n_+,n_-}(\{\omega\}) = \int D(\psi, \bar{\psi}) \int_{U(N)} dU \exp \left( -\bar{\psi}_{+a}^k (\delta^{kl} - e^{i\omega_{+a}} U^{kl}) \psi_{+a}^l - \bar{\psi}_{-b}^k (\delta^{kl} - e^{-i\omega_{-b}} \bar{U}^{lk}) \psi_{-b}^l \right). \quad (2)$$

In the case of Hamiltonian systems, the next steps of Efetov's method are (i) to average over the Gaussian disorder, thereby producing a term quartic in the  $\psi$  fields, (ii) to decouple this term by the introduction of a Hubbard-Stratonovich supermatrix field  $Q$ , and (iii) to make a saddle-point approximation eliminating the so-called massive modes. The final outcome of this convoluted and mathematically very delicate procedure is an integral over fields that live on a nonlinear supermanifold, called Efetov's  $\sigma$ -model space with unitary symmetry. It is difficult to see how this procedure could be adapted to the present case as the Haar integral over  $U(N)$  is very far from being Gaussian. Fortunately, there is another way to proceed.

The key innovation made here is to trade the integral over  $U(N)$  *directly* for an integral over Efetov's  $\sigma$ -model space with unitary symmetry. This is achieved by the following remarkable identity:

$$\begin{aligned} & \int_{U(N)} dU \exp \left( \bar{\psi}_{+a}^i U^{ij} \psi_{+a}^j + \bar{\psi}_{-b}^j \bar{U}^{ij} \psi_{-b}^i \right) \\ &= \int D\mu_N(Z, \tilde{Z}) \exp \left( \bar{\psi}_{+a}^i Z_{ab} \psi_{-b}^i + \bar{\psi}_{-b}^j \tilde{Z}_{ba} \psi_{+a}^j \right), \end{aligned} \quad (3)$$

which can be proved by letting the Lie supergroup  $Gl(nN|nN)$  act on a Fock space of bosons and fermions and using some basic concepts of the theory of generalized coherent states. The proof is given in detail in the Appendix. Here we confine ourselves to spelling out the meaning of the right-hand side. The new integration variables  $Z = \{Z_{ab}\}$  and  $\tilde{Z} = \{\tilde{Z}_{ab}\}$  are supermatrices whose Boson-Fermion block decomposition is written

$$Z = \begin{pmatrix} Z_{BB} & Z_{BF} \\ Z_{FB} & Z_{FF} \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} \tilde{Z}_{BB} & \tilde{Z}_{BF} \\ \tilde{Z}_{FB} & \tilde{Z}_{FF} \end{pmatrix}.$$

Each block of  $Z$  ( $\tilde{Z}$ ) is a rectangular matrix of dimension  $n_+ \times n_-$  ( $n_- \times n_+$ ). The integration measure is

$$D\mu_N(Z, \tilde{Z}) = D(Z, \tilde{Z}) \text{SDet}(1 - \tilde{Z}Z)^N,$$

where  $D(Z, \tilde{Z})$  denotes the flat Berezin measure on the space of supermatrices  $Z, \tilde{Z}$ ,<sup>1</sup> and  $\text{SDet}$  is the superdeterminant. The domain of integration is defined by the conditions

$$\tilde{Z}_{BB} = Z_{BB}^\dagger, \quad \tilde{Z}_{FF} = -Z_{FF}^\dagger,$$

and the requirement that all eigenvalues of the positive hermitian  $n_- \times n_-$  matrix  $\tilde{Z}_{BB}Z_{BB}$  be less than unity. The integration measure is normalized by  $\int D\mu_N(Z, \tilde{Z}) = 1$ . In the next subsection we will argue that  $Z$  and  $\tilde{Z}$  should be interpreted as parameterizing Efetov's unitary  $\sigma$ -model space.

Let me mention in passing that integrals of a type similar to the left-hand side of (3) appear in lattice gauge theory, where  $\psi$  and  $U$  represent the quark and gluon fields, respectively. Motivated by this similarity, one might refer to the upper index of  $\psi$  as "color" and the lower one as "flavor". We could then say that (3) integrates out the gluon fields, which carry color, and replaces them by an integral over color-singlet meson fields carrying flavor. Note that the color degrees of freedom of the tensor  $\psi$  are *uncoupled* on the right-hand side.

By using the generalized Hubbard-Stratonovitch transformation (3) with  $\bar{\psi}_{+a}^k \rightarrow \bar{\psi}_{+a}^k e^{i\omega_{+a}}$  and  $\bar{\psi}_{-b}^k \rightarrow \bar{\psi}_{-b}^k e^{-i\omega_{-b}}$ , we can process the expression (2) as follows:

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<sup>1</sup>To be precise, I should say that this is true only *locally*. Berezin measures typically suffer from a global anomaly [11,12]. What determines  $D(Z, \tilde{Z})$  and its anomaly uniquely (up to multiplication by a constant) is the requirement of invariance under translations on Efetov's  $\sigma$ -model space  $G/H$ , see Sec. II B and the Appendix.

$$\begin{aligned}
F_{n_+, n_-}(\{\omega\}) &= \int D\mu_N(Z, \tilde{Z}) \int D(\psi, \bar{\psi}) \exp \left( -\bar{\psi}_{+a}^k \psi_{+a}^k - \bar{\psi}_{-b}^k \psi_{-b}^k \right. \\
&\quad \left. + \bar{\psi}_{+a}^k e^{i\omega_{+a}} Z_{ab} \psi_{-b}^k + \bar{\psi}_{-b}^k e^{-i\omega_{-b}} \tilde{Z}_{ba} \psi_{+a}^k \right) \\
&= \int D\mu_N(Z, \tilde{Z}) \text{SDet}^{-N} \begin{pmatrix} 1 & -e^{i\omega_{+}} Z \\ -e^{-i\omega_{-}} \tilde{Z} & 1 \end{pmatrix} \\
&= \int D\mu_N(Z, \tilde{Z}) \text{SDet}^{-N} (1 - \tilde{Z} e^{i\omega_{+}} Z e^{-i\omega_{-}}) \\
&= \int D(Z, \tilde{Z}) \text{SDet}^{-N} (1 - (1 - \tilde{Z} Z)^{-1} \tilde{Z} (e^{i\omega_{+}} Z e^{-i\omega_{-}} - Z)). \tag{4}
\end{aligned}$$

In the first step we did the Gaussian superintegral over  $\psi, \bar{\psi}$ , producing a superdeterminant. As the different species labeled by  $i = 1, \dots, N$  are uncoupled, this superdeterminant separates into a product of  $N$  factors. The expression  $e^{i\omega_{+}} Z$  stands for the supermatrix with matrix elements  $e^{i\omega_{+a}} Z_{ab}$ . The second step in (4) is immediate from the elementary identity

$$\text{SDet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{SDet}(A) \text{SDet}(D - CA^{-1}B).$$

The third step follows from the formula  $D\mu_N(Z, \tilde{Z}) = D(Z, \tilde{Z}) \text{SDet}^N(1 - \tilde{Z}Z)$  along with the multiplicative property of the superdeterminant:  $\text{SDet}(A) \text{SDet}(B) = \text{SDet}(AB)$ .

Let me emphasize that the result (4) is *exact for all  $N$* . In contrast with Efetov's method, *no saddle-point approximation was used, and no massive modes had to be eliminated*. Equation (4) reduces the integral (1) over the  $N^2$  real freedoms of  $U(N)$  to an integral over the  $4n_+n_-$  complex freedoms of the supermatrices  $Z, \tilde{Z}$ . Clearly, such a scheme is efficient when  $N$  is large and  $n_+n_-$  is small.

## B. Geometric meaning of $Z, \tilde{Z}$

We start with a concise mathematical description of Efetov's  $\sigma$ -model space of unitary symmetry. Let  $G = \text{Gl}(n|n)$ , the complex Lie supergroup of regular supermatrices of dimension  $(n+n) \times (n+n)$ , and put  $n = n_A + n_R$ . (Later we shall make the identifications  $n_A = n_+$  and  $n_R = n_-$ .) Using tensor-product notation, we take  $\Sigma_z \in \text{Gl}(n|n)$  to be the diagonal matrix  $\Sigma_z = 1_{1|1} \otimes \text{diag}(1_{n_A}, -1_{n_R})$  where  $1_{n_A}$  is the  $n_A$ -dimensional unit matrix, and  $1_{1|1}$  is the  $(1+1)$ -dimensional unit matrix in Boson-Fermion space. Matrices  $h \in \text{Gl}(n|n)$  that commute with  $\Sigma_z$  are of the form

$$h = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix}, \quad \text{if } \Sigma_z = \begin{pmatrix} 1_{n_A|n_A} & 0 \\ 0 & -1_{n_R|n_R} \end{pmatrix}$$

is the matrix presentation of  $\Sigma_z$ . Thus the condition  $h\Sigma_z = \Sigma_z h$  (or, equivalently,  $h = \Sigma_z h \Sigma_z$ ) fixes a subgroup  $H = \text{Gl}(n_A|n_A) \times \text{Gl}(n_R|n_R)$  of  $G$ . The coset space  $G/H$  is a complex-analytic supermanifold in the sense of Berezin-Kostant-Leites [13,14], and has complex dimension  $d_{\text{Boson}} = d_{\text{Fermion}} = 2n_A n_R$ . The canonical projection  $G \rightarrow G/H$  endows the coset space  $G/H$  with a natural  $G$ -invariant geometry. If  $G/H$  is modeled by supermatrices  $Q = g\Sigma_z g^{-1}$ , this geometry is given by the rank-two tensor field  $\text{STr}(dQ)^2$ , where  $\text{STr}$  denotes the supertrace. The  $G$ -invariant Berezin integration measure on  $G/H$  is denoted by  $Dg_H$ , or  $DQ$ . The objects one wants to integrate [12] are the holomorphic functions on  $G/H$ , i.e. functions with a holomorphic dependence on a set of complex local coordinates of the complex-analytic supermanifold  $G/H$ . Such functions are written  $f(gH)$  or  $F(Q)$ . The domain of integration is taken to be a Riemannian submanifold  $M_B \times M_F$  of the support of  $G/H$ , where

$$\begin{aligned}
M_B &= U(n_A, n_R)/U(n_A) \times U(n_R) & (\text{BB sector}), \\
M_F &= U(n_A + n_R)/U(n_A) \times U(n_R) & (\text{FF sector}).
\end{aligned}$$

In this way one gets a  $G$ -invariant Berezin integral

$$\int_{M_B \times M_F} Dg_H f(gH) = \int_{M_B \times M_F} Dg_H f(g_0 g H) \quad (g_0 \in G), \tag{5}$$

which is called the integral over Efetov's  $\sigma$ -model space with unitary symmetry. Translation  $gH \mapsto g_0 g H$  by an element  $g_0 \in G$  does not leave the integration domain  $M_B \times M_F$  invariant, in general. Nevertheless, if  $f$  is holomorphic, as

is assumed, (5) does hold because Cauchy's theorem applies and allows to undo the deformation of the integration domain. In the terminology of [12] the pair  $(G/H, M_B \times M_F)$  is called a Riemannian symmetric superspace of type AIII|AIII. (Nine more types exist.)

To make contact with Sec. II A we introduce a suitable parameterization of  $G/H$ . Using the above presentation where  $\Sigma_z = \text{diag}(1_{n_A|n_A}, -1_{n_R|n_R})$  we decompose  $g \in G$  as

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Since  $h = \text{diag}(h_+, h_-)$ , the invariants for the right action of  $h \in H$  on  $G$  are  $Z := BD^{-1}$  and  $\tilde{Z} := CA^{-1}$ . We may take the matrix elements of  $Z, \tilde{Z}$  for a set of complex local coordinates of the coset space  $G/H$ . The expression for  $Q = g\Sigma_z g^{-1}$  in these coordinates is

$$Q = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1}. \quad (6)$$

From the expression for the invariant rank-two tensor field,

$$\text{STr}(dQ)^2 = \text{const} \times \text{STr}(1 - \tilde{Z}Z)^{-1} d\tilde{Z}(1 - Z\tilde{Z})^{-1} dZ,$$

one easily finds the  $G$ -invariant Berezin measure  $DQ$  in these coordinates to be the (locally) flat one,  $D(Z, \tilde{Z})$  [15]. The flatness results from cancellations due to supersymmetry.

To make the restriction to the Riemannian submanifold  $M_B \times M_F$ , note that for a unitary matrix  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{U}(n_A + n_R)$ , we have  $b^\dagger = -(d - ca^{-1}b)^{-1}ca^{-1}$  and  $d^\dagger = (d - ca^{-1}b)^{-1}$ . This implies

$$Z_{\text{FF}}^\dagger = (bd^{-1})^\dagger = d^{-1\dagger}b^\dagger = -ca^{-1} = -\tilde{Z}_{\text{FF}}.$$

For the noncompact case  $g \in \text{U}(n_A, n_R)$ , the expression for  $b^\dagger$  has its sign reversed, so

$$Z_{\text{BB}}^\dagger = d^{-1\dagger}b^\dagger = +ca^{-1} = +\tilde{Z}_{\text{BB}}.$$

Moreover, the pseudo-unitarity of  $g \in \text{U}(n_A, n_R)$  implies that  $a^\dagger a - c^\dagger c = 1$ , from which we infer

$$\tilde{Z}_{\text{BB}}Z_{\text{BB}} = ca^{-1}a^{-1\dagger}c^\dagger = c(1 + c^\dagger c)^{-1}c^\dagger,$$

so that all eigenvalues of the positive hermitian  $n_R \times n_R$  matrix  $\tilde{Z}_{\text{BB}}Z_{\text{BB}}$  must be less than unity.

All these facts suggest to put  $n_A \equiv n_+$ ,  $n_R \equiv n_-$  and identify the supermatrices  $Z, \tilde{Z}$  defined here with those figuring in the generalized Hubbard-Stratonovitch transformation (3). This identification is made rigorous in the Appendix, where a Lie-algebraic proof of (3) is given. The proof relies in an essential way on the existence of the invariant integral (5), and on the identification of  $D(Z, \tilde{Z})$  with the  $G$ -invariant Berezin measure  $Dg_H$  on  $G/H$ .

For further orientation, consider the simplest example  $n_+ = n_- = 1$ , where  $M_B = \text{U}(1, 1)/\text{U}(1) \times \text{U}(1) \simeq \text{H}^2$  (two-hyperboloid) and  $M_F = \text{U}(2)/\text{U}(1) \times \text{U}(1) \simeq \text{S}^2$  (two-sphere). In this case the BB and FF blocks of  $Z$  are just numbers. The expression for  $Z_{\text{FF}}$  in terms of the usual polar ( $\theta$ ) and azimuthal ( $\varphi$ ) angles on  $\text{S}^2$  turns out [16] to be  $Z_{\text{FF}} = \tan(\theta/2) \exp i\varphi$ . From this it is easily seen that  $Z_{\text{FF}}$  can be interpreted as being a complex stereographic coordinate for  $\text{S}^2$ . Its BB analog can be parameterized in terms of the hyperbolic polar angle  $\theta_B$  on  $\text{H}^2$  by  $Z_{\text{BB}} = \tanh(\theta_B/2) \exp i\varphi_B$ .

To conclude this subsection, let us recast the result (4) in the coordinate-free language developed here. From (6) or

$$Q = g\Sigma_z g^{-1} = \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} (1 - Z\tilde{Z})^{-1} & -Z(1 - \tilde{Z}Z)^{-1} \\ \tilde{Z}(1 - Z\tilde{Z})^{-1} & -(1 - \tilde{Z}Z)^{-1} \end{pmatrix} =: \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix},$$

we read off the relations

$$\begin{aligned} \tilde{Z}(1 - Z\tilde{Z})^{-1} &= Q_{-+}/2, & (1 - Z\tilde{Z})^{-1} &= (1 + Q_{++})/2, \\ Z &= -(1 + Q_{++})^{-1}Q_{+-}, & (1 - \tilde{Z}Z)^{-1} &= (1 - Q_{--})/2. \end{aligned}$$

Inserting these into (4) we can express the generating function as a  $Q$ -integral:

$$F_{n_+, n_-}(\{\omega\}) = \int DQ \text{SDet}^{-N} (1 - Q_{--} + Q_{-+}e^{i\omega_+}(1 + Q_{++})^{-1}Q_{+-}e^{-i\omega_-}). \quad (7)$$

Let me repeat that this expression is exact for all  $N$ .

The following remark may be helpful. It is well known [17] that the density of states of the *Gaussian* Unitary Ensemble tends to a semicircle in the large- $N$  limit. It is also known [18] that the so-called “massive modes”, which show up in Efetov’s method and are eliminated by a saddle-point approximation, are needed to reproduce the *nonuniformity* of the semicircular density of states. In contradistinction, the eigenvalues of a unitary matrix  $U \in \text{CUE}$  are uniformly distributed over the unit circle, on average. Qualitatively speaking, it is this *uniformity* of the CUE spectrum that allows a  $Q$ -integral representation to be derived without introducing the massive modes and without making any saddle-point approximation.

### C. Large- $N$ Limit

For further calculations the formula (4) turns out to be most convenient. In the large- $N$  limit the integrand in (4) substantially differs from zero only when the difference of matrices  $\omega_+ - \omega_-$  is of order  $1/N$ . This fact allows us to expand:

$$e^{i\omega_+} Z e^{-i\omega_-} - Z = i(\omega_+ Z - Z \omega_-) + \mathcal{O}(1/N^2),$$

which leads to the approximate equality

$$F_{n_+, n_-}(\{\omega\}) \simeq \int D(Z, \tilde{Z}) \exp iN \text{STr} \left( \omega_+ Z \tilde{Z} (1 - Z \tilde{Z})^{-1} - \omega_- \tilde{Z} Z (1 - \tilde{Z} Z)^{-1} \right).$$

The  $N$  eigenvalues of the unitary matrix  $U$  are distributed over the unit circle with circumference  $2\pi$ , which results in the mean density of eigenphases being  $\nu = N/2\pi$ . By setting  $\omega := \text{diag}(\omega_+, \omega_-)$  we can write the integrand in the form  $\exp iN \text{STr} \omega (Q - \Sigma_z)/2$ . Thus, the large- $N$  limit of the generating function  $F_{n_+, n_-}(\omega \equiv \{\omega\})$  evaluated on a scale set by the mean spacing, is

$$\lim_{N \rightarrow \infty} F_{n_+, n_-}(2\pi\omega/N) = \int dQ \exp i\pi \text{STr} \omega (Q - \Sigma_z).$$

Equivalently, we may write  $Q = g \Sigma_z g^{-1}$ , and get

$$\lim_{N \rightarrow \infty} F_{n_+, n_-}(2\pi\omega/N) = \int_{M_B \times M_F} Dg_H \exp i\pi \text{STr} \omega (g \Sigma_z g^{-1} - \Sigma_z). \quad (8)$$

As is well known [2,18,12], the same expression is obtained for the Gaussian Unitary Ensemble, which confirms the expected equivalence [10] of the Circular and Gaussian ensembles in the large- $N$  limit.

We are now going to calculate from the large- $N$  result (8) the so-called  $n$ -level correlation function  $R_n$  for any  $n$ . (Efetov [2] calculated only the two-level correlation function.) To that end, we will use a supersymmetric variant of a result known in the physics literature as the Itzykson-Zuber formula [19], which is actually a special case of a more general result proved 23 years earlier by the mathematician Harish-Chandra.

Let  $K$  be any connected compact semisimple Lie group, which we assume to be realized by  $n \times n$  matrices  $k$ , and let  $\mathcal{T}$  be a maximal commuting (or Cartan) subalgebra of  $\text{Lie}(K)$ . Without loss we take  $\mathcal{T}$  to be the diagonal matrices in  $\text{Lie}(K)$ . If  $A, B$  are two elements of  $\mathcal{T}$ , Harish-Chandra showed that (see Theorem 2 of [20])

$$\int_K dk \exp \text{Tr} A k B k^{-1} = \frac{\text{const}}{p(A)p(B)} \sum_{\hat{s} \in W[K]} (-1)^{|\hat{s}|} \exp \text{Tr} A \hat{s} B, \quad (9)$$

where  $p(A) = \prod_{\alpha > 0} \alpha(A)$  is the product over a set of positive roots  $\alpha$  of the pair  $[\mathcal{T}, \text{Lie}(K)]$ , and  $\text{const}$  is a normalization constant. The sum runs over the Weyl group  $W[K]$ , which is the discrete group of inequivalent transformations  $\mathcal{T} \rightarrow \mathcal{T}$  by  $A \mapsto \hat{s}A := sAs^{-1}$  with  $s \in K$ . The Weyl group can be considered as a subgroup of the symmetric group  $S_n$  acting on the elements of the diagonal matrices  $A \in \mathcal{T}$ . The symbol  $|\hat{s}| = 0, 1$  denotes the parity of  $\hat{s}$ . For the special case  $K = \text{SU}(n)$  (or  $\text{U}(n)$ , it makes no real difference) one has  $p(A) = \prod_{i < j} (A_i - A_j)$ , where  $A_i$  are the elements of the diagonal matrix  $A$ , and the Weyl group coincides with  $S_n$ , i.e.  $(\hat{s}A)_i = (sAs^{-1})_i =: A_{\hat{s}(i)}$  where  $\hat{s} \in S_n$  is a permutation and  $|\hat{s}|$  is the parity of that permutation. The general result (9) thus takes the more familiar form [19]

$$\int_{U(n)} dU \exp \text{Tr} A U B U^{-1} = \text{const} \times \prod_{i < j} (A_i - A_j)^{-1} (B_i - B_j)^{-1} \times \text{Det} (e^{A_i B_j})_{i,j=1,\dots,n}.$$

The right-hand side of (9) can be viewed as the stationary-phase approximation to the integral on the left-hand side, with the points of stationarity being enumerated by the elements  $\hat{s} \in W[K]$ . Thus, Harish-Chandra's formula states that the stationary-phase approximation is *exact* in this case. M. Stone [21] has traced this remarkable coincidence to a hidden symmetry of the integral.

I conjecture that the formula (9) extends to any compact classical Lie *supergroup*  $K$ :

$$\int_K Dk \exp \text{STr} A k B k^{-1} = \frac{\text{const}}{p(A)p(B)} \sum_{\hat{s} \in W[K_0]} (-1)^{|\hat{s}|} \exp \text{STr} A \hat{s} B, \quad (10)$$

where  $W[K_0]$  now is the Weyl group of the ordinary group  $K_0$  supporting  $K$ , and  $p(A)$  turns into the product of positive bosonic roots divided by the product of positive fermionic roots. The strategy of the proof ought to be a supersymmetric extension of that of Harish-Chandra and use the theory of invariant differential operators [20,22,23]. For the special case of the unitary Lie supergroup  $K = U(n|n)$ , which is the one we will need below, two proofs using alternative strategies can be found in Refs. [24] and [25]. In that case, if  $A = \text{diag}(A_{1,B}, \dots, A_{n,B}; A_{1,F}, \dots, A_{n,F})$ , one has

$$p(A) = \prod_{i < j} (A_{i,B} - A_{j,B})(A_{i,F} - A_{j,F}) / \prod_{i,j} (A_{i,B} - A_{j,F}), \quad (11)$$

and  $W[K_0] = W[U(n) \times U(n)] \simeq S_n \times S_n$ , so

$$\sum_{\hat{s} \in W[K_0]} (-1)^{|\hat{s}|} \exp \text{STr} A \hat{s} B = \text{Det} (e^{A_{i,B} B_{j,B}})_{i,j=1,\dots,n} \times \text{Det} (e^{-A_{i,F} B_{j,F}})_{i,j=1,\dots,n}.$$

In a reduced form tailored to physical applications, the Harish-Chandra-Itzykson-Zuber integral for  $U(n|n)$  made its first appearance in [26].

Now recall that the  $n$ -level correlation function  $R_n(\theta_1, \theta_2, \dots, \theta_n)$  is defined [17] as the probability density to find, given a level (i.e. an eigenphase) at  $\theta_1$ ,  $n - 1$  levels at positions  $\theta_2, \dots, \theta_n$ , irrespective of the positions of all other levels. By making use of the identities

$$\begin{aligned} \frac{\partial}{\partial \varphi} \Big|_{\varphi=\theta} \frac{\text{Det}(1 - e^{i\varphi} U)}{\text{Det}(1 - e^{i\theta} U)} &= i \text{Tr} (1 - e^{-i\theta} U^\dagger)^{-1}, \\ \frac{e^{-\varepsilon} \varepsilon / \pi}{(1 - e^{-\varepsilon - i\theta})(1 - e^{-\varepsilon + i\theta})} &= \frac{\varepsilon / \pi}{2(1 - \cos \theta) + 4 \sinh^2(\varepsilon/2)} \xrightarrow{\varepsilon \rightarrow 0} \delta(\theta), \end{aligned}$$

we can extract  $R_n$  from the generating function  $F_{n,n}$ , Eq. (1), by the following formula:

$$R_n(\theta_1, \dots, \theta_n) = \lim_{\varepsilon \rightarrow 0} \left( \frac{\varepsilon}{\pi} \right)^n \prod_{\alpha=1}^n \frac{\partial^2}{\partial \varphi_{+\alpha} \partial \varphi_{-\alpha}} \Big|_{\varphi_{\pm\alpha} = \theta_{\alpha} \pm i\varepsilon} F_{n,n}(\{\theta_{\alpha} + i\varepsilon, \theta_{\alpha} - i\varepsilon, \varphi_{+\alpha}, \varphi_{-\alpha}\}), \quad (12)$$

which serves as the starting point of our supersymmetric calculation of  $R_n$ .

The first step toward doing the integral (8) is to introduce Efetov's polar coordinates [2] on the coset space  $G/H$ :

$$\begin{aligned} gH &= haH, \quad \text{where} \quad h = \begin{pmatrix} h_+ & 0 \\ 0 & h_- \end{pmatrix} \in H, \\ \text{and} \quad a &= \exp \begin{pmatrix} 0 & z \\ z & 0 \end{pmatrix}, \quad z = \text{diag}(x_1, \dots, x_n, iy_1, \dots, iy_n). \end{aligned}$$

The variables  $x_i, y_j$  contained in  $a$  are the “radial” coordinates, and the supermatrix  $h$  is parameterized by a set of “angular” coordinates. The choice of integration domain  $M_B \times M_F$  for  $G/H$  dictates that we must integrate over *real*  $x_i, y_j$  and superunitary  $h \in U(n|n) \times U(n|n) \subset H$ . The polar-coordinate form of the invariant integral (5) is

$$\int Dg_H f(gH) = \int \left( \int Dh f(haH) \right) J(a) da + \dots, \quad (13)$$

where  $Dh$  is the Haar-Berezin measure of  $H$ , and  $da$  denotes the Euclidean measure on the abelian group of radial elements  $a$ . The Jacobian of the transformation to polar coordinates is given [27,28] by  $J(a)da =$

$$\begin{aligned} & \frac{\prod_{i < j} \sinh^2(x_i - x_j) \sinh^2(x_i + x_j) \sin^2(y_i - y_j) \sin^2(y_i + y_j)}{\prod_{i,j} \sinh^2(x_i + iy_j) \sinh^2(x_i - iy_j)} \prod_{k=1}^n \sinh(2x_k) \sin(2y_k) dx_k dy_k \\ &= \text{const} \times \frac{\prod_{i < j} (\cosh 2x_i - \cosh 2x_j)^2 (\cos 2y_i - \cos 2y_j)^2}{\prod_{i,j} (\cosh 2x_i - \cos 2y_j)^2} \prod_{k=1}^n d(\cosh 2x_k) d(\cos 2y_k). \end{aligned} \quad (14)$$

For the second equality sign the trigonometric identity

$$2 \sinh(\alpha + \beta) \sinh(\alpha - \beta) = \cosh 2\alpha - \cosh 2\beta$$

was used. The integration domain is taken to be  $0 \leq x_k < \infty$ ,  $0 \leq y_k \leq \pi/2$ . The dots in (13) indicate correction terms that arise from the anomaly of the invariant Berezin measure in polar coordinates. The form of these corrections was investigated and completely determined in [29]. Their main characteristic is that they are supported on the *boundary* of the integration domain (which is why they are alternatively known as “boundary terms”). More precisely, they are obtained by setting  $x_i = y_j = 0$  for one or several pairs  $(x_i, y_j)$ . It will turn out that these boundary terms do not contribute to  $R_n$ , Eq. (12), and therefore do not need to be specified explicitly for our purposes.

The next step is to express  $\text{STr} \omega g \Sigma_z g^{-1}$  in the exponential of the integrand of (8) in terms of polar coordinates:

$$\begin{aligned} \text{STr} \omega g \Sigma_z g^{-1} &= \text{STr} \Sigma_z \omega h a^2 h^{-1} = \text{STr} \Sigma_z \omega h \cosh(2 \ln a) h^{-1} \\ &= \text{STr} \omega_+ h_+ A h_+^{-1} - \text{STr} \omega_- h_- A h_-^{-1}, \end{aligned}$$

where  $A = \cosh 2z$ ,  $\omega = \text{diag}(\omega_+, \omega_-)$ , and  $\omega_{\pm} = \text{diag}(\theta_1 \pm i\varepsilon, \dots, \theta_n \pm i\varepsilon; \varphi_{\pm 1}, \dots, \varphi_{\pm n})$ . By this and  $Dh = Dh_+ Dh_-$  the angular integral over  $U(n|n) \times U(n|n)$  factors into a product of two integrals over  $U(n|n)$ . By formula (10) for  $U(n|n)$ , the integral over  $h_+$  gives

$$\int_{U(n|n)} Dh_+ \exp i \text{STr} \omega_+ h_+ A h_+^{-1} = \frac{\text{const}}{p(\omega_+) p(A)} \sum_{\hat{s}} (-1)^{|\hat{s}|} \exp i \text{STr} \omega_+ \hat{s} A.$$

A similar expression results from doing the integral over  $h_-$ . Multiplying both integrals we get a factor  $p(A)^{-2}$ . By (11) and  $A = \text{diag}(\cosh 2x_1, \dots, \cosh 2x_n; \cos 2y_1, \dots, \cos 2y_n)$  this factor exactly cancels the factor multiplying  $dA := \prod_k d(\cosh 2x_k) d(\cos 2y_k)$  in (14). Hence we obtain

$$\begin{aligned} & \int Dg_H \exp i \text{STr} \omega g \Sigma_z g^{-1} \\ &= \frac{\text{const}}{p(\omega_+) p(\omega_-)} \sum_{\hat{s}_+, \hat{s}_-} (-1)^{|\hat{s}_+| + |\hat{s}_-|} \int dA \exp i \text{STr} A (\hat{s}_+ \omega_+ - \hat{s}_- \omega_-) \\ & \quad + \text{boundary terms.} \end{aligned}$$

The inverse of the product of positive roots  $p(\omega_+)^{-1}$  vanishes linearly with each difference  $\omega_{+\alpha, B} - \omega_{+\alpha, F} = \theta_{\alpha} + i\varepsilon - \varphi_{+\alpha}$ . An analogous statement holds for  $p(\omega_-)^{-1}$ . Therefore, on taking the  $2n$  first derivatives at  $\varphi_{\pm\alpha} = \theta_{\alpha} \pm i\varepsilon$  [see (12)], we simply obtain a constant, which can be absorbed into the normalization. Note  $\text{STr} \Sigma_z \omega|_{\varphi_{\pm\alpha} = \theta_{\alpha} \pm i\varepsilon} = 0$ .

A further simplification results from the fact that  $dA$  and the integration domain for  $A$  are Weyl-invariant. This permits us to reduce the double sum over  $\hat{s}_+, \hat{s}_-$  to a single sum over the *relative* permutation  $\hat{s} := \hat{s}_-^{-1} \hat{s}_+$ , times an overall factor  $\dim W[U(n) \times U(n)] = \dim(S_n \times S_n) = n!^2$ . Absorbing this constant factor into the normalization, we obtain for the  $n$ -level correlation function the following intermediate result:

$$R_n = \text{const} \times \sum_{\hat{s}} (-1)^{|\hat{s}|} \lim_{\varepsilon \rightarrow 0} \varepsilon^n \int dA \exp i \text{STr} A \left( (\hat{\theta} + i\varepsilon) - \hat{s}(\hat{\theta} - i\varepsilon) \right),$$

where  $\hat{\theta} = \text{diag}(\theta_B, \theta_F)$ ,  $\theta_B = \theta_F = \text{diag}(\theta_1, \dots, \theta_n)$ . The integral factors into a product of  $2n$  one-dimensional exponential integrals over the elements of the diagonal matrix  $A$ ,  $n$  of which are compact ( $-1 \leq \cos 2y_i \leq 1$ ), and the other  $n$  are noncompact ( $1 \leq \cosh 2x_i < \infty$ ). The Weyl group element  $\hat{s} =: (\hat{s}_B, \hat{s}_F) \in W[U(n) \times U(n)]$  acts on the BB sector by  $\hat{s}_B$  and on the FF sector by  $\hat{s}_F$ . If  $\theta_i = (\hat{s}_B \theta)_i$ , the integral over  $t_i = \cosh 2x_i$  diverges as  $\varepsilon^{-1}$  in the limit  $\varepsilon \rightarrow 0$ . The maximal divergence occurs for  $\hat{s}_B = \text{identity}$ , in which case a singular factor  $\varepsilon^{-n}$  is produced, canceling

the prefactor  $\varepsilon^n$  and producing a finite result in the limit  $\varepsilon \rightarrow 0$ . The boundary terms mentioned earlier disappear at this stage as they contain at most  $n-1$  noncompact integrations and therefore diverge more weakly than  $\varepsilon^{-n}$ . What remains are the  $n$  compact integrations over  $t_i = \cos 2y_i$ . Using the elementary integral  $\int_{-1}^{+1} dt \exp i\pi\theta t = 2 \sin(\pi\theta)/\pi\theta$  we immediately arrive at the final result:

$$\lim_{N \rightarrow \infty} R_n(2\pi\theta_1/N, \dots, 2\pi\theta_n/N) = \sum_{\hat{s} \in W[U(n)]} (-1)^{|\hat{s}|} \prod_{i=1}^n \frac{\sin \pi(\theta_i - \hat{s}\theta_i)}{\pi(\theta_i - \hat{s}\theta_i)} = \text{Det} \left( \frac{\sin \pi(\theta_i - \theta_j)}{\pi(\theta_i - \theta_j)} \right)_{i,j=1, \dots, n}, \quad (15)$$

where the correct value of the normalization constant was restored by hand. This result coincides with the one obtained by the Dyson-Mehta orthogonal polynomial method [17] in the large- $N$  limit. In Sec. IV we are going to see that the first form of the result (15) extends in a very simple way to the circular ensemble that is obtained by taking instead of the unitary group the symplectic one. In that sense, this is the “good” way of writing the result for  $R_n$ .

### III. CIRCULAR ORTHOGONAL ENSEMBLE

If an open quantum mechanical system possesses an anti-unitary symmetry  $\mathcal{T}$  (time reversal, for example) and  $\mathcal{T}^2 = +1$ , there exists a basis of scattering states such that the scattering matrix  $S$  is symmetric:  $S = S^T$ . The set of symmetric  $S$ -matrices can be parameterized in terms of the unitary matrices  $U \in U(N)$  by  $S = UU^T$ . The product  $S = UU^T$  is invariant under right multiplication of  $U$  by any orthogonal matrix, which means [30] that  $S = UU^T$  lives on the coset space  $U(N)/O(N)$ .

The Circular Orthogonal Ensemble (COE) of random-matrix theory is defined [17] by taking  $S$  to be distributed according to the uniform measure on  $U(N)/O(N)$ . We denote this measure by  $d\mu(S)$ . The generating function for COE spectral correlators,  $F_{n+,n-}$ , is defined as in (1) but with the substitutions  $U(N) \rightarrow U(N)/O(N)$ ,  $U \rightarrow S$ , and  $dU \rightarrow d\mu(S)$ . Because the invariant measure  $d\mu(S)$  is induced by the Haar measure  $dU$  of  $U(N)$  through the projection  $U(N) \rightarrow U(N)/O(N)$ , we can write the defining expression for  $F_{n+,n-}$  also as follows:

$$F_{n+,n-}(\{\theta, \varphi\}) = \int_{U(N)} dU \prod_{\alpha=1}^{n+} \frac{\text{Det}(1 - e^{i\varphi+\alpha} UU^T)}{\text{Det}(1 - e^{i\theta+\alpha} UU^T)} \prod_{\beta=1}^{n-} \frac{\text{Det}(1 - e^{-i\varphi-\beta} \bar{U} U^\dagger)}{\text{Det}(1 - e^{-i\theta-\beta} \bar{U} U^\dagger)}.$$

This expression will now be processed by the method of Sec. II. To do so we use the trick of doubling the dimension:

$$\text{Det}(1 - e^{i\gamma} UU^T) = \text{Det} \begin{pmatrix} 1 & e^{i\gamma} U \\ U^T & 1 \end{pmatrix}.$$

The extra degree of freedom implied by this doubling will be called “quasispin” and denoted by the symbols  $\uparrow, \downarrow$ .

After the introduction of quasispin, we express  $F_{n+,n-}$  as a Gaussian superintegral in the usual way, see Sec. II A, Eq. (2). As before we put  $\{\omega\} = \{\theta, \varphi\}$ . Then we lump the two terms containing  $U$  into a single one:

$$\bar{\psi}_{+a\uparrow}^i e^{i\omega+a} U^{ij} \psi_{+a\downarrow}^j + \bar{\psi}_{+a\downarrow}^j (U^T)^{ji} \psi_{+a\uparrow}^i \equiv \phi_{+A}^i U^{ij} \chi_{+A}^j,$$

where the tensors  $\phi_+$  and  $\chi_+$  have components

$$\phi_{+A}^i = \{\phi_{+a1}^i, \phi_{+a2}^i\} = \{\bar{\psi}_{+a\uparrow}^i, \psi_{+a\uparrow}^i (-1)^{|a|}\}, \quad \chi_{+A}^j = \{\chi_{+a1}^j, \chi_{+a2}^j\} = \{e^{i\omega+a} \psi_{+a\downarrow}^j, \bar{\psi}_{+a\downarrow}^j\}.$$

Here  $|a| = 0$  if  $a = (\alpha, B)$ , and  $|a| = 1$  if  $a = (\alpha, F)$ . The terms involving  $U^\dagger$  are manipulated in a similar manner:

$$\bar{\psi}_{-b\uparrow}^i \bar{U}^{ij} \psi_{-b\downarrow}^j + \bar{\psi}_{-b\downarrow}^j e^{-i\omega-b} (U^\dagger)^{ji} \psi_{-b\uparrow}^i \equiv \chi_{-B}^j \bar{U}^{ij} \phi_{-B}^i, \quad \text{where} \\ \chi_{-B}^j = \{\chi_{-b1}^j, \chi_{-b2}^j\} = \{\bar{\psi}_{-b\downarrow}^j e^{-i\omega-b}, \psi_{-b\downarrow}^j (-1)^{|b|}\}, \quad \phi_{-B}^i = \{\phi_{-b1}^i, \phi_{-b2}^i\} = \{\psi_{-b\uparrow}^i, \bar{\psi}_{-b\uparrow}^i\}.$$

In the next step we apply the generalized Hubbard-Stratonovitch transformation (3):

$$\int dU \exp \left( \phi_{+A}^i U^{ij} \chi_{+A}^j + \chi_{-B}^j \bar{U}^{ij} \phi_{-B}^i \right) = \int D\mu_N(Z, \tilde{Z}) \exp \left( \phi_{+A}^i Z_{AB} \phi_{-B}^i + \chi_{-B}^j \tilde{Z}_{BA} \chi_{+A}^j \right).$$

Note that the exponential on the right-hand side separates into terms containing  $\phi$  and  $\chi$ . The same is true for the terms  $\bar{\psi}\psi$  that do not involve  $U$ . Therefore the integral splits into two integrals, one over the tensor  $\phi$  and the

other one over the tensor  $\chi$ . Consider the  $\phi$  integral first, and define an orthogonal matrix  $\tau_+$  (“charge conjugation”) by requiring the tensor  $(\tau_+)_{AA'}\phi_{+A'}^i$  to have components  $\{\psi_{+a\uparrow}^i, \bar{\psi}_{+a\uparrow}^i\}$ . The matrix that does this job is given in quasispin block decomposition by

$$\tau_+ = \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix},$$

where  $\sigma$  is the matrix for superparity ( $\sigma_{aa'} = (-1)^{|a|}\delta_{aa'}$ ). The same matrix acting in the space of “negative” indices (with the appropriate change of dimension) is denoted by  $\tau_-$ . Having introduced these matrices we can write

$$\bar{\psi}_{+a\uparrow}^i \psi_{+a\uparrow}^i = \frac{1}{2} \phi_{+A}^i (\tau_+)_{AA'} \phi_{+A'}^i, \quad \text{and} \quad \bar{\psi}_{-b\uparrow}^i \psi_{-b\uparrow}^i = \frac{1}{2} \phi_{-B}^i (\tau_-)_{BB'} \phi_{-B'}^i.$$

On collecting terms we encounter the integral

$$\begin{aligned} & \int D\phi \exp \left( \phi_{+A}^i Z_{AB} \phi_{-B}^i - \frac{1}{2} \phi_{+A}^i (\tau_+)_{AA'} \phi_{+A'}^i - \frac{1}{2} \phi_{-B}^i (\tau_-)_{BB'} \phi_{-B'}^i \right) \\ &= (-1)^{nN/2} \text{SDet}^{-N/2} \begin{pmatrix} \tau_+ & -Z \\ -Z^T \tau_- & \tau_- \end{pmatrix} = \text{SDet}^{-N/2} (1 - Z^T \tau_+ Z \tau_-^{-1}). \end{aligned}$$

Here  $(Z^T)_{BA} = Z_{AB}(-1)^{(|A|+1)|B|}$  is the supertranspose, with  $|A| = 0$  if  $A = +(\alpha, B) \uparrow$ , and  $|A| = 1$  if  $A = +(\alpha, F) \uparrow$ . The  $\chi$  integral is done in a similar fashion and gives  $\text{SDet}^{-N/2} (1 - \tilde{Z} \tau_+^{-1} e^{i\omega_+} \tilde{Z}^T e^{-i\omega_-} \tau_-)$ . By making the variable substitution

$$Z \mapsto e^{i\omega_+/4} i Z e^{-i\omega_-/4}, \quad \tilde{Z} \mapsto -e^{i\omega_-/4} i \tilde{Z} e^{i\omega_+/4},$$

we bring the final result into the symmetrical form

$$\begin{aligned} F_{n_+, n_-}(\{\omega\}) &= \int D\mu_N(Z, \tilde{Z}) \text{SDet}^{-N/2} \left( 1 + Z^T \tau_+ e^{i\omega_+/2} Z e^{-i\omega_-/2} \tau_-^{-1} \right) \\ &\quad \times \text{SDet}^{-N/2} \left( 1 + \tilde{Z} \tau_+^{-1} e^{i\omega_+/2} \tilde{Z}^T e^{-i\omega_-/2} \tau_- \right), \end{aligned}$$

which is exact for all  $N$ . The matrices  $Z, \tilde{Z}$  parameterize the complex coset space  $\text{Gl}(2n|2n)/\text{Gl}(2n_+|2n_+) \times \text{Gl}(2n_-|2n_-)$ .

We turn to the large- $N$  limit. In order for this limit to be nontrivial, we must again take the difference  $\omega_+ - \omega_-$  to be of order  $1/N$ . By  $D\mu_N(Z, \tilde{Z}) = D(Z, \tilde{Z}) \exp N \text{STr} \ln(1 - \tilde{Z} Z)$  the integrand at  $\omega_+ = \omega_- = \text{const} \times \text{identity}$  can be written as

$$\exp \left( N \text{STr} \ln(1 - \tilde{Z} Z) - \frac{N}{2} \text{STr} \ln(1 + Z^T \tau_+ Z \tau_-^{-1}) - \frac{N}{2} \text{STr} \ln(1 + \tilde{Z} \tau_+^{-1} \tilde{Z}^T \tau_-) \right).$$

For large  $N$ , the integral is dominated by contributions from the subspace determined by the equation

$$Z = -\tau_+^{-1} \tilde{Z}^T \tau_-.$$

This subspace is referred to as the saddle-point manifold. Integration over the Gaussian fluctuations normal to the saddle-point manifold gives unity by supersymmetry, in the large- $N$  limit.

By restricting the integrand to the saddle-point manifold, expanding with respect to  $\omega_+ - \omega_-$  and keeping only the terms that survive for  $N \rightarrow \infty$ , we obtain

$$\lim_{N \rightarrow \infty} F_{n_+, n_-}(2\pi\{\omega\}/N) = \int D(Z, \tilde{Z}) \exp i\pi \text{STr} \left( \omega_+ Z \tilde{Z} (1 - Z \tilde{Z})^{-1} - \omega_- \tilde{Z} Z (1 - \tilde{Z} Z)^{-1} \right).$$

This has the same form as for the CUE, except for the replacement  $N \rightarrow N/2$ , the doubling of the dimensions of the supermatrices  $Z, \tilde{Z}$ , and the imposition of the constraint  $Z = -\tau_+^{-1} \tilde{Z}^T \tau_-$ . On putting  $\omega \equiv \{\omega\} = \text{diag}(\omega_+, \omega_-)$  we can re-express the result in the form

$$\lim_{N \rightarrow \infty} F_{n_+, n_-}(2\pi\omega/N) = \int_{M_B \times M_F} Dg_H \exp i\pi \text{STr} \omega (g \Sigma_z g^{-1} - \Sigma_z)/2.$$

The relation between  $Q = g\Sigma_z g^{-1}$  and  $Z, \tilde{Z}$  is formally the same as before. The constraint  $Z = -\tau_+^{-1} \tilde{Z}^T \tau_-$  is known [18,12] to reduce the complex superspace  $\text{Gl}(2n|2n)/\text{Gl}(2n_+|2n_+) \times \text{Gl}(2n_-|2n_-)$  to the complex submanifold  $\text{Osp}(2n|2n)/\text{Osp}(2n_+|2n_+) \times \text{Osp}(2n_-|2n_-)$ . The conjugation relations  $\tilde{Z}_{\text{BB}} = Z_{\text{BB}}^\dagger$  and  $\tilde{Z}_{\text{FF}} = -Z_{\text{FF}}^\dagger$  translate into

$$M_{\text{B}} = \text{SO}(2n_+, 2n_-)/\text{SO}(2n_+) \times \text{SO}(2n_-), \quad M_{\text{F}} = \text{Sp}(2n_+ + 2n_-)/\text{Sp}(2n_+) \times \text{Sp}(2n_-).$$

For  $n_+ = n_- = 1$ , the above expression for the generating function  $F_{n_+, n_-}$  can be calculated by introducing Efetov's polar coordinates. The resulting formula for the two-level correlation function  $R_2$  coincides with that of [2]. It is not clear at present how to do this calculation for general  $n_+, n_-$ . [Harish-Chandra's formula does *not* extend to this case as neither  $\Sigma_z$  nor  $\omega = \text{diag}(\omega_+, \omega_-)$  are elements of  $\text{Lie}(\text{Osp}(2n|2n))$ .]

#### IV. CIRCULAR ENSEMBLE OF TYPE C

Dyson's Circular Ensembles COE and CSE are constructed by starting from the unitary group and passing to the coset spaces  $\text{U}(N)/\text{O}(N)$  and  $\text{U}(2N)/\text{Sp}(2N)$ . Other circular ensembles of relevance [31] to mesoscopic physics can be obtained by taking for the starting point the orthogonal or symplectic group instead of the unitary one. Again the option of forming various coset spaces exists. Of these possibilities the technically simplest one is to consider an ensemble of scattering matrices  $S$  drawn at random from the symplectic group  $\text{Sp}(2N)$ , with no coset projection done. This is what we shall do in the present section. The probability distribution for  $S$  will be taken to be the Haar measure  $dS$  of  $\text{Sp}(2N)$ . The circular ensemble so defined is called "type C". A physical system where such an ensemble can be realized is a chaotic Andreev quantum dot [32] with time-reversal symmetry broken by a weak magnetic field.

The defining equations of  $\text{Sp}(2N)$  are

$$S^{-1\dagger} = S = \mathcal{C} S^{-1T} \mathcal{C}^{-1}, \quad \text{where} \quad \mathcal{C} = i\sigma_y \otimes 1_N = \begin{pmatrix} 0 & 1_N \\ -1_N & 0 \end{pmatrix}$$

is the symplectic unit in  $2N$  dimensions. The polar (or Cartan) decomposition of an element  $S \in \text{Sp}(2N)$  has the form  $S = k e^{i\hat{\theta}} k^{-1}$  where  $\hat{\theta} = \sigma_z \otimes \text{diag}(\theta_1, \dots, \theta_N)$  and  $\sigma_z = \text{diag}(+1, -1)$ . Thus if  $\theta_1$  is an eigenphase of  $S$ , then so is  $-\theta_1$ . This symmetry under reflection  $\theta \rightarrow -\theta$  is called a "particle-hole" symmetry.

All information about the eigenphase correlations of the circular random-matrix ensemble of type C is contained in the generating function

$$F_n(\{\theta, \varphi\}) = \int_{\text{Sp}(2N)} dS \prod_{\alpha=1}^n \frac{\text{Det}(1 - e^{i\varphi_\alpha} S)}{\text{Det}(1 - e^{i\theta_\alpha} S)} \quad (\text{Im}\theta_\alpha > 0).$$

This expression is simpler than the corresponding one for the CUE because  $\text{Det}(1 - zS) = \text{Det}(1 - zS^\dagger)$  as a result of the particle-hole symmetry. By putting  $\{\omega_a\} = \{\theta_\alpha, \varphi_\alpha\}$  and proceeding as in Sec. II A, we get a Gaussian superintegral representation for  $F_n$ :

$$F_n(\{\omega\}) = \int D(\psi, \bar{\psi}) \int_{\text{Sp}(2N)} dS \exp -\bar{\psi}_a^i (\delta^{ij} - e^{i\omega_a} U^{ij}) \psi_a^j.$$

The components  $\psi_a^i, \bar{\psi}_a^i$  of the tensors  $\psi, \bar{\psi}$  are bosonic for  $a = (\alpha, \text{B})$  and fermionic for  $a = (\alpha, \text{F})$ . The analog of (3) reads:

$$\int_{\text{Sp}(2N)} dS \exp \bar{\psi}_a^i S^{ij} \psi_a^j = \int_{M_{\text{B}} \times M_{\text{F}}} D\mu_N(Z, \tilde{Z}) \exp \left( \frac{1}{2} \bar{\psi}_a^i Z_{ab} \mathcal{C}^{ij} \bar{\psi}_b^j - \frac{1}{2} \psi_a^i (-1)^{|a|} \tilde{Z}_{ab} \mathcal{C}^{ij} \psi_b^j \right).$$

As before,  $D\mu_N(Z, \tilde{Z}) = D(Z, \tilde{Z}) \text{SDet}(1 - \tilde{Z}Z)^N$ , but  $Z$  and  $\tilde{Z}$  are now *square* supermatrices and are subject to the symmetry conditions

$$Z_{ab} = (-1)^{|a|+|b|+|a||b|+1} Z_{ba}, \quad \tilde{Z}_{ab} = (-1)^{|a||b|+1} \tilde{Z}_{ba}.$$

These conditions can be succinctly written as

$$Z = -Z^T \sigma, \quad \tilde{Z} = -\sigma \tilde{Z}^T,$$

where  $\sigma$  is the superparity as before. They express the fact [12] that  $Z, \tilde{Z}$  parameterize the complex supermanifold  $G/H = \text{Osp}(2n|2n)/\text{Gl}(n|n)$ , see also below. The integration domain  $M_B \times M_F$  is fixed by

$$\tilde{Z}_{BB} = Z_{BB}^\dagger, \quad \tilde{Z}_{FF} = -Z_{FF}^\dagger,$$

and the requirement that all eigenvalues of the positive hermitian  $n \times n$  matrix  $\tilde{Z}_{BB} Z_{BB}$  be less than unity. It can be shown [12] that this means

$$M_B = \text{SO}^*(2n)/\text{U}(n), \quad M_F = \text{Sp}(2n)/\text{U}(n).$$

$(G/H, M_B \times M_F)$  is called a Riemannian symmetric superspace of type  $DIII|CI$ .

In the simplest case  $n = 1$  the supermatrices  $Z, \tilde{Z}$  have the form

$$Z = \begin{pmatrix} 0 & \zeta_1 \\ \zeta_1 & z_1 \end{pmatrix}, \quad \tilde{Z} = \begin{pmatrix} 0 & \zeta_2 \\ -\zeta_2 & z_2 \end{pmatrix},$$

where  $z_1, z_2$  are complex commuting and  $\zeta_1, \zeta_2$  are complex anticommuting numbers. It is seen that the BB sector  $M_B$  is trivial in this case (in fact  $\text{SO}^*(2)/\text{U}(1)$  consists of just a single point). The variable  $z_2 = -\bar{z}_1$  of the FF sector can be interpreted as the complex stereographic coordinate of a two-sphere  $S^2 \simeq \text{Sp}(2)/\text{U}(1)$ . It is very important to note [12] that the variables  $\zeta_1, \zeta_2$  are *not related by any kind of complex conjugation*.

The proof of the generalized Hubbard-Stratonovitch identity for the present case is closely analogous to that for the CUE, presented in detail in the Appendix. (To understand the following remark you must study that appendix first.) The key idea is to consider the generalized coherent states

$$\begin{aligned} |Z\rangle &:= \exp\left(\frac{1}{2}\bar{c}_a^i Z_{ab} C^{ij} \bar{c}_b^j\right) |0\rangle \text{SDet}(1 - \tilde{Z}Z)^{N/2}, \\ \langle Z| &:= \text{SDet}(1 - \tilde{Z}Z)^{N/2} \langle 0| \exp\left(\frac{1}{2}c_a^i (-1)^{|a|} \tilde{Z}_{ab} C^{ij} c_b^j\right), \end{aligned}$$

built from an absolute vacuum  $c_a^i |0\rangle = 0$  by repeatedly acting with Bose and Fermi creation and annihilation operators obeying the supercommutation relations  $[c_a^i, \bar{c}_b^j] = \delta^{ij} \delta_{ab}$ , and to exploit the fact that  $P = \int D(Z, \tilde{Z}) |Z\rangle \langle Z|$  projects on the singlet sector of the symplectic group acting on Fock space by  $\bar{c}_a^i \mapsto S^{ij} \bar{c}_a^j$ .

Use of the generalized Hubbard-Stratonovitch transformation leads to

$$\begin{aligned} F_n(\{\omega\}) &= \int D\mu_N(Z, \tilde{Z}) \text{SDet}^{-N} \begin{pmatrix} 1 & -e^{i\omega} Z e^{i\omega} \\ -\tilde{Z} & 1 \end{pmatrix} \\ &= \int D(Z, \tilde{Z}) \text{SDet}^{-N} \left(1 - (1 - \tilde{Z}Z)^{-1} \tilde{Z} (e^{i\omega} Z e^{i\omega} - Z)\right), \end{aligned}$$

where  $\omega$  stands for the diagonal matrix with elements  $\omega_a$ . To arrive at an equivalent  $Q$ -integral representation, we let  $g$  run through the orthosymplectic Lie supergroup  $G = \text{Osp}(2n|2n)$  defined by

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix}^{-1} \begin{pmatrix} 0 & \sigma \\ 1 & 0 \end{pmatrix},$$

and set  $Z = BD^{-1}$ ,  $\tilde{Z} = CA^{-1}$ ,

$$Q := g \Sigma_z g^{-1} := \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & Z \\ \tilde{Z} & 1 \end{pmatrix}^{-1} =: \begin{pmatrix} Q_{++} & Q_{+-} \\ Q_{-+} & Q_{--} \end{pmatrix},$$

as before. The subgroup  $H \subset G$  of elements  $h = \text{diag}(A, A^{-1T})$  satisfying  $h \Sigma_z h^{-1} = \Sigma_z$  is isomorphic to  $\text{Gl}(n|n)$ . Hence the elements of the supermatrix  $Q$  are functions on the coset space  $G/H = \text{Osp}(2n|2n)/\text{Gl}(n|n)$ . By simple manipulations we obtain

$$F_n(\{\omega\}) = \int DQ \text{SDet}^{-N} \left(1 - Q_{--} + Q_{-+} e^{i\omega} (1 + Q_{++})^{-1} Q_{+-} e^{i\omega}\right),$$

which is to be compared with (7).

In the large- $N$  limit the exact expression for  $F_n$  simplifies to

$$\lim_{N \rightarrow \infty} F_n(\pi\{\omega\}/N) = \int_{M_B \times M_F} Dg_H \exp i\pi \text{STr} \hat{\omega} (g \Sigma_z g^{-1} - \Sigma_z)/2,$$

where  $\hat{\omega} = \text{diag}(\omega, -\omega)$ . Note that this is formally identical to (8). The CUE  $n$ -level correlation function  $R_n$  was extracted from that equation in Sec. II C. It turns out that the corresponding calculation can be done in the present case, too, by extending Harish-Chandra's formula (9) to the orthosymplectic Lie supergroup. [In contrast with the situation for the COE, the matrices  $\hat{\omega}$  and  $\Sigma_z$  now *are* elements of  $\text{Lie}(\text{Osp}(2n|2n))$ .] To keep the present paper within size, let me omit the details of this calculation (which in any case is very similar to that of Sec. II C) and simply state the result for  $R_n$  thus obtained:

$$\lim_{N \rightarrow \infty} R_n(\pi\theta_1/N, \dots, \pi\theta_n/N) = \sum_{\hat{s} \in W[\text{Sp}(2n)]} (-1)^{|\hat{s}|} \prod_{i=1}^n \frac{\sin \pi(\theta_i - \hat{s}\theta_i)}{\pi(\theta_i - \hat{s}\theta_i)}.$$

The Weyl group  $W[\text{Sp}(2n)]$  is generated by the operations of transposition  $\theta_{i-1} \leftrightarrow \theta_i$  ( $i = 2, \dots, n$ ) and reflection  $\theta_1 \mapsto -\theta_1$ , all of which have odd parity  $|\hat{s}| = 1$ . Please observe the perfect analogy to (15): the Weyl group of  $U(n)$  has simply been replaced by the Weyl group of  $\text{Sp}(2n)$ . Specialization to  $n = 1, 2$  gives

$$\begin{aligned} \lim_{N \rightarrow \infty} R_1(\pi\theta_1/N) &= 1 - \frac{\sin 2\pi\theta_1}{2\pi\theta_1}, \\ \lim_{N \rightarrow \infty} R_2(\pi\theta_1/N, \pi\theta_2/N) &= \left(1 - \frac{\sin 2\pi\theta_1}{2\pi\theta_1}\right) \left(1 - \frac{\sin 2\pi\theta_2}{2\pi\theta_2}\right) \\ &\quad - \left(\frac{\sin \pi(\theta_1 - \theta_2)}{\pi(\theta_1 - \theta_2)} - \frac{\sin \pi(\theta_1 + \theta_2)}{\pi(\theta_1 + \theta_2)}\right)^2. \end{aligned}$$

Precisely the same expressions were obtained for the correlation functions of the *Gaussian* random-matrix ensemble of type *C* in [31]. The reasoning used there was heuristic and took recourse to the mapping on a one-dimensional free Fermi gas with Dirichlet boundary conditions at the origin  $\theta = 0$ . This coincidence of results is of course expected from the large- $N$  equivalence of the Circular and Gaussian Ensembles.

## V. SUMMARY AND OUTLOOK

The key result of this paper is the Hubbard-Stratonovitch identity (3), relating an integral over the unitary group  $U(N)$  to an integral over a Riemannian symmetric superspace of type  $AIII|AIII$  (Efetov's unitary  $\sigma$ -model space). The detailed proof given in the Appendix derives this identity from the standard properties of generalized coherent states. Instead of  $U(N)$  one can also consider the symplectic group  $\text{Sp}(2N)$ , or the orthogonal group in an even number of dimensions,  $\text{SO}(2N)$ . In these cases, identities similar to (3) exist and relate the corresponding group integrals to integrals over Riemannian symmetric superspaces of type  $DIII|CI$  and  $CI|DIII$  respectively. For  $\text{Sp}(2N)$  this was described briefly in Sec. IV. The case of  $\text{SO}(2N)$  was not discussed here and is left for future work.

In the present paper the generalized Hubbard-Stratonovitch transformation was applied to Dyson's Circular Unitary Ensemble (CUE, or type *A*) and the circular ensemble of type *C*. In both cases the large- $N$  limit of the  $n$ -level correlation function was calculated for all  $n$ , by appropriate extensions of Harish-Chandra's formula. The method is not restricted to spectral correlations but can be used for wave amplitude correlations and transport coefficients as well. It can also be adapted to other symmetry classes. The trick that works for Dyson's Circular Orthogonal Ensemble (COE, or type *AI*) is to write the elements  $S$  of the COE in the form  $S = UU^T$  with  $U \in \text{CUE}$  and then proceed as in the unitary case. A similar trick works for Dyson's Circular Symplectic Ensemble (type *AII*), the Circular Ensemble of type *CI* and, in fact, any one of the large class [12] of circular ensembles.

As was mentioned in the introduction, nontrivial applications to random network models are expected. What puts these within reach is the fact that the Hubbard-Stratonovitch scheme (3) is valid for all  $N$ , including  $N = 1$ . This permits to transform a network model with random  $U(1)$  phases into a theory of coupled supermatrices  $Z, \tilde{Z}$ . Doing so for the Chalker-Coddington model [5], for example, and taking a continuum limit, one obtains [9] the two-dimensional nonlinear  $\sigma$  model augmented by Pruisken's topological term.

Another realm of fruitful application of the identity (3) will be quantum chaotic maps. Andreev et al. [4] have recently argued that energy averaging for deterministic Hamiltonian systems can be used to derive a "ballistic" [3] nonlinear  $\sigma$  model, in much the same way as impurity averaging for disordered systems leads to the usual "diffusive"  $\sigma$  model. When one goes from Hamiltonian systems to periodically driven ones or maps, the role of the Hamiltonian passes to a unitary operator  $U$ , the Floquet or time-evolution operator. As the spectrum of  $U$  lies on the unit circle in

$\mathbb{C}$ , an approach analogous to that of Andreev et al. will employ U(1)-phase averaging instead of energy averaging. It is clear that, by carrying out this phase average with the help of the identity (3), one will be able to derive a nonlinear  $\sigma$  model for maps. Work in this direction is in progress [33].

Finally, let me mention that the identity (3) is not restricted to the supersymmetric case. Similar formulas can also be derived rigorously when the spinor  $\psi$  is purely bosonic or fermionic. (In the bosonic case, convergence of all integrals places a lower bound on the allowed values of  $N$ .) The fermionic version of (3) is particularly exciting as it promises nontrivial applications to lattice gauge theory, by enabling an exact transformation from the gauge degrees of freedom to meson fields. (Note that the usual plaquette term  $\text{Tr}UUUU$  of lattice gauge theory can be generated by coupling a number of heavy fermionic ghosts to the gauge field via  $\bar{\psi}U\psi$ .) In this way, one may succeed in constructing a nonperturbative proof of a result due to Witten [34], who summed planar diagrams to argue that large- $N$  quantum chromodynamics is equivalent to a weakly coupled theory of mesons, in the low-energy limit. Sadly, although the exact gluon-meson transformation works for the gauge groups  $U(N)$  and  $\text{Sp}(2N)$  (and, presumably, for  $\text{SO}(2N)$  as well), it does not easily extend to  $\text{SU}(N)$ , except for  $N = 2$ , where one has the accidental isomorphism  $\text{SU}(2) \simeq \text{Sp}(2)$ .

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## VI. APPENDIX

Given a compact Lie group  $K$  and a unitary irreducible action  $T$  of  $K$  on a module  $V$  with lowest (or highest) weight  $|0\rangle$ , one can consider objects of the form  $T_k|0\rangle$  ( $k \in K$ ). These are called *generalized coherent states*, and they have many nice properties [35–37]. Most importantly, they resolve the unit operator on  $V$ :

$$1_V = \int_K dk T_k|0\rangle\langle 0|T_k^\dagger,$$

where  $dk$  denotes the Haar measure of  $K$ , normalized by  $\int_K dk |\langle 0|T_k|0\rangle|^2 = 1$ . A well-known application of this is the derivation of coherent-state path integrals for quantum spin systems, in which case  $K = \text{SU}(2)$  and  $V$  is a spin- $S$  representation space of  $\text{SU}(2)$ . The objective of the present appendix is to extend some of the mathematics of generalized coherent states to the case where instead of  $K$  we have the supergroup  $G = \text{Gl}(n|n)$  that acts on Efetov’s complexified coset space  $G/H$  of unitary symmetry. This will eventually yield an easy proof of the identity (3).

We start by defining a group action of  $\text{Gl}(n|n)$  on a Fock space of bosons and fermions and developing various useful structures that come with it. The general context and our notations have been laid down in Secs. II A, II B. Recall that the tensor  $\psi$  has components  $\psi_{+a}^i, \psi_{-b}^i$ , where  $i = 1, \dots, N$ , and  $a = (\alpha, \sigma)$ ,  $b = (\beta, \sigma)$  are composite indices with range  $\alpha = 1, \dots, n_+$ ;  $\beta = 1, \dots, n_-$ ; and  $\sigma = \text{B, F}$  (Bosons and Fermions). For notational convenience, we introduce a composite index  $A$  taking values  $+a$  or  $-b$ . Thus,  $\{\psi_A^i\} = \{\psi_{+a}^i, \psi_{-b}^i\}$ . The terminology we use is to refer to the upper index  $i$  as “color” and the lower index  $A$  as “flavor”. Now we introduce creation and annihilation operators  $\bar{c}_A^i, c_A^i$  for bosons and fermions, which obey the canonical supercommutation relations

$$[c_A^i, \bar{c}_B^j] := c_A^i \bar{c}_B^j - (-1)^{|A||B|} \bar{c}_B^j c_A^i = \delta_{AB} \delta^{ij},$$

where  $|A| = 0$  if  $A = \pm(\alpha, \text{B})$  and  $|A| = 1$  if  $A = \pm(\alpha, \text{F})$ . These operators act on a Fock space with vacuum  $|0\rangle$  defined by  $c_A^i|0\rangle = 0$ . They can be viewed as the quantum mechanical counterparts of the “classical” variables  $\bar{\psi}_A^i, \psi_A^i$ .

For the following it is very convenient (though not necessary) to make the transformation

$$c_{-b}^i \mapsto \bar{c}_{-b}^i, \quad \bar{c}_{-b}^i \mapsto (-1)^{|b|+1} c_{-b}^i,$$

which is canonical, i.e. leaves the supercommutation relations unchanged. (The operators  $c_{+a}^i, \bar{c}_{+a}^i$  are not transformed.) If the Fock space is endowed with its usual inner product (so that  $\bar{c}_A^i$  is adjoint to  $c_A^i$ ), the transformation is *nonunitary* in the bosonic sector since, if  $b, b^\dagger$  are adjoints of each other, the transformed canonical pair  $b^\dagger, -b$  no longer are. One can check that *no mathematical problems are caused by this nonunitarity*. After the transformation, the Fock space vacuum  $|0\rangle$  and its dual  $\langle 0|$  are annihilated by

$$c_{+a}^i|0\rangle = \bar{c}_{-b}^j|0\rangle = 0 \quad \text{and} \quad \langle 0|\bar{c}_{+a}^i = \langle 0|c_{-b}^j = 0.$$

We may imagine that the “positive-energy states” ( $+a$ ) are empty while the “negative-energy states” ( $-b$ ) have been filled. Of course, for unitary bosons no such thing as a filled negative-energy sea exists, which is another way of seeing why the above canonical transformation must be nonunitary. Note also

$$\sum_{\beta=1}^{n_-} \bar{c}_{-\beta,F}^i c_{-\beta,F}^j |0\rangle = |0\rangle \times (+\delta^{ij} n_-), \quad \sum_{\beta=1}^{n_-} \bar{c}_{-\beta,B}^i c_{-\beta,B}^j |0\rangle = |0\rangle \times (-\delta^{ij} n_-).$$

(The summation convention has been temporarily suspended for clarity.)

Consider now the set of operators  $E_{AB}^{ij} := \bar{c}_A^i c_B^j$ . By the commutation relations for the  $c, \bar{c}$  these bilinears form a  $\mathfrak{gl}(nN, nN)$  Lie superalgebra ( $n = n_+ + n_-$ ):<sup>2</sup>

$$\begin{aligned} [E_{AB}^{ij}, E_{CD}^{kl}] &:= E_{AB}^{ij} E_{CD}^{kl} - (-1)^{(|A|+|B|)(|C|+|D|)} E_{CD}^{kl} E_{AB}^{ij} \\ &= \delta^{jk} \delta_{BC} E_{AD}^{il} - (-1)^{(|A|+|B|)(|C|+|D|)} \delta^{li} \delta_{DA} E_{CB}^{kj}. \end{aligned}$$

Two subalgebras we shall have use for are the  $\mathfrak{gl}(N)$  Lie algebra generated by the flavor-singlet operators  $\sum_A \bar{c}_A^i c_A^i$ , and the  $\mathfrak{gl}(n, n)$  Lie superalgebra generated by the color-singlet operators  $\sum_i \bar{c}_A^i c_B^i$ . These subalgebras commute and, moreover, they are *maximal* relative to each other, i.e. the maximal subalgebra of  $\mathfrak{gl}(nN, nN)$  whose elements commute with all elements of  $\mathfrak{gl}(N)$ , is  $\mathfrak{gl}(n, n)$ , and vice versa.

In the sequel, we focus attention on that part of Fock space where all  $\mathfrak{gl}(N)$  generators vanish identically:

$$\sum_A \bar{c}_A^i c_A^j |\text{flavor state}\rangle \equiv 0.$$

This subspace contains the vacuum:

$$\sum_A \bar{c}_A^i c_A^j |0\rangle = \sum_{\beta=1}^{n_-} \left( \bar{c}_{-\beta,B}^i c_{-\beta,B}^j + \bar{c}_{-\beta,F}^i c_{-\beta,F}^j \right) |0\rangle = |0\rangle \times \delta^{ij} (-n_- + n_-) = 0,$$

and will be called the “flavor sector” (or color-neutral sector). We claim that the action of  $\mathfrak{gl}(n, n)$  on the flavor sector is *irreducible*, i.e. every state of the flavor sector can be reached by a multiple action of the flavor operators  $\sum_i \bar{c}_A^i c_B^i$  on the vacuum. The argument goes as follows. By employing the occupation number representation of Fock space, one easily sees that  $\mathfrak{gl}(nN, nN)$  acts irreducibly on the subspace of Fock space selected by the condition  $\sum_{A,i} \bar{c}_A^i c_A^i = 0$ , which contains the color-neutral sector and might be called the “charge-neutral” sector. Thus every charge-neutral state  $|\mathcal{N}\rangle$  can be obtained by acting on the vacuum (or lowest-weight) state  $|0\rangle$  with the raising operators,  $\bar{c}_{+a}^i c_{-b}^j$ , of  $\mathfrak{gl}(nN, nN)$ :

$$|\mathcal{N}\rangle = \sum F_{a_1 b_1 \dots a_r b_r}^{i_1 j_1 \dots i_r j_r} \bar{c}_{+a_1}^{i_1} c_{-b_1}^{j_1} \dots \bar{c}_{+a_r}^{i_r} c_{-b_r}^{j_r} |0\rangle.$$

If  $R \in \text{Gl}(N)$  is a rotation in color space, such a state transforms as a number of copies of the vector representation  $\bar{c}_A^i \mapsto \sum_j R^{ij} \bar{c}_A^j$  and the same number of copies of the co-vector representation  $c_A^i \mapsto \sum_j (R^{-1})^{ji} c_A^j$ . Now assume  $|\mathcal{N}\rangle$  to be color-neutral, i.e.  $\sum_A \bar{c}_A^i c_A^i |\mathcal{N}\rangle = 0$ . In order for this equation to hold, the indices of the expansion coefficients have to be contracted pairwise ( $\pi$  here stands for permutations):

$$F_{a_1 b_1 \dots a_r b_r}^{i_1 j_1 \dots i_r j_r} = \sum_{\pi \in S_r} f(a_1 b_1 \dots a_r b_r; \pi) \delta_{i_1 \pi(j_1)} \dots \delta_{i_r \pi(j_r)},$$

by a standard result on the reduction of tensor-product representations of  $\text{Gl}(N)$ . This form of the expansion coefficients permits to express the operators that create  $|\mathcal{N}\rangle$  out of the vacuum, in terms of the raising operators  $\sum_i \bar{c}_{+a}^i c_{-b}^i$  of the flavor superalgebra  $\mathfrak{gl}(n, n)$ . Thus the vacuum  $|0\rangle$  is a cyclic vector for the action of  $\mathfrak{gl}(n, n)$  on the flavor sector. Because the action of every particle-hole creation operator  $\sum_i \bar{c}_{+a}^i c_{-b}^i$  can be undone by the corresponding annihilation operator  $\sum_i \bar{c}_{-b}^i c_{+a}^i$ , the existence of a cyclic vector implies irreducibility. (q.e.d.)

In addition to the Lie superalgebra  $\mathfrak{gl}(n, n)$  we shall need a corresponding Lie supergroup,  $\text{Gl}(n|n)$ . Recall that  $G := \text{Gl}(n|n)$  is defined as the group of regular complex supermatrices of dimension  $(n+n) \times (n+n)$ .<sup>3</sup> We associate with each element  $g \in G$  an operator  $T_g$  on Fock space by

<sup>2</sup>One advantage gained by the nonunitary canonical transformation  $c_{-b}^i \mapsto \bar{c}_{-b}^i$ ,  $\bar{c}_{-b}^i \mapsto (-1)^{|b|+1} c_{-b}^i$  is that it allows us to present all  $\mathfrak{gl}(nN, nN)$  generators in the form  $\bar{c}c$ , whereas in the original presentation they would have been of various types  $\bar{c}c$ ,  $\bar{c}\bar{c}$  and  $cc$ .

<sup>3</sup>The change in notation from the Lie superalgebra  $\mathfrak{gl}(n, n)$  (komma) to the Lie supergroup  $\text{Gl}(n|n)$  (vertical bar) reminds us of the fact that the definition of the latter requires the specification of some parameter Grassmann algebra, whereas the former does not.

$$T_g = \exp \left( \sum_{A,B,i} \bar{c}_A^i (\ln g)_{AB} c_B^i \right).$$

In view of the multivaluedness of the logarithm, we need to demonstrate that this operator is well-defined. As the diagonalizable supermatrices are dense in  $G$ , it suffices to do so for an element  $g \in G$  of the form  $g = S\lambda S^{-1}$ , where  $\lambda$  is a diagonal matrix containing the eigenvalues of  $g$ . The eigenvalues of  $\ln g = S(\ln \lambda)S^{-1}$  are defined up to permutations and shifts by integer multiples of  $2\pi i$ . We follow the convention that the Grassmann variables in the supermatrix  $\ln g$  anticommute with the fermionic operators of the set  $\{\bar{c}_A^i, c_B^j\}$ . It is then obvious that the transformation  $c_A^i \mapsto \gamma_A^i = \sum_B (S^{-1})_{AB} c_B^i$ ,  $\bar{c}_A^i \mapsto \bar{\gamma}_A^i = \sum_B \bar{c}_B^i S_{BA}$  is canonical. Note  $\sum \bar{c}_A^i (\ln g)_{AB} c_B^i = \sum \bar{\gamma}_A^i \gamma_A^i \ln \lambda_A$ . Now the number operators  $\hat{n}_A^i = \bar{\gamma}_A^i \gamma_A^i$  have integer eigenvalues, so the ambiguity  $\ln \lambda_A \rightarrow \ln \lambda_A + 2\pi i n$  is unobservable in  $T_g = \exp \left( \sum_{A,i} \bar{\gamma}_A^i \gamma_A^i \ln \lambda_A \right)$ , and  $T_g$  is indeed well-defined.

We next claim that the mapping  $g \mapsto T_g$  is a homomorphism of Lie supergroups:

$$T_g T_h = T_{gh}.$$

The proof goes as follows. First, we use  $T_{\exp X}$  to transform  $\bar{c}_A^i$  by

$$\bar{c}_A^i \mapsto T_{\exp X} \bar{c}_A^i T_{\exp X}^{-1} = \exp \text{ad}(\bar{c}_B^j X_{BC} c_C^j) \bar{c}_A^i = \bar{c}_B^i (\exp X)_{BA},$$

where  $\text{ad}(\hat{A})\hat{B} := \hat{A}\hat{B} - \hat{B}\hat{A}$  stands for the commutator, and the summation convention is now back in force. On setting  $\exp X = g$  we get  $T_g \bar{c}_A^i T_g^{-1} = \bar{c}_B^i g_{BA}$  and

$$T_g T_h \bar{c}_A^i T_h^{-1} T_g^{-1} = T_g (\bar{c}_B^i h_{BA}) T_g^{-1} = \bar{c}_C^i g_{CB} h_{BA} = \bar{c}_B^i (gh)_{BA} = T_{gh} \bar{c}_A^i T_{gh}^{-1},$$

from which we infer that the product  $T_{gh}^{-1} T_g T_h$  commutes with  $\bar{c}_A^i$ . For identical reasons,  $T_{gh}^{-1} T_g T_h$  commutes also with  $c_B^j$ . Now, because the set of Fock operators  $\{\bar{c}_A^i, c_B^j\}$  act irreducibly on Fock space, the vanishing commutator implies that  $T_{gh}^{-1} T_g T_h$  is a multiple of the unit operator:

$$T_{gh}^{-1} T_g T_h = f(g, h) \times \mathbf{1}.$$

The second step is to show  $f(g, h) \equiv 1$ . For that we use a formula [22] for the differential of the exponential map of any Lie algebra onto its Lie group:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} e^{X+t\dot{X}} &= e^X \mathcal{T}_X(\dot{X}), \quad \text{where} \\ \mathcal{T}_X &= \sum_{n=0}^{\infty} (-1)^n \frac{\text{ad}^n(X)}{(n+1)!} = \frac{1 - e^{-\text{ad}(X)}}{\text{ad}(X)}. \end{aligned}$$

This formula trivially extends to the case at hand, namely  $\mathfrak{gl}(n|n)$ , a Lie algebra with Grassmann structure [38]. The set where  $\mathcal{T}_X$  has an inverse is dense in  $\mathfrak{gl}(n|n)$ . We put  $g = \exp X$ ,  $h(t) = \exp tY$ , and apply the formula for the differential twice, the first time in the reverse order:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} T_{\exp(X) \exp(tY)} &= \left. \frac{d}{dt} \right|_{t=0} T_{\exp(X + \mathcal{T}_X^{-1}(tY))} \\ &= \left. \frac{d}{dt} \right|_{t=0} \exp \bar{c}_A^i (X + \mathcal{T}_X^{-1}(tY))_{AB} c_B^i \\ &= \exp (\bar{c}_A^i X_{AB} c_B^i) \left. \frac{d}{dt} \right|_{t=0} \exp \bar{c}_A^i (\mathcal{T}_X \circ \mathcal{T}_X^{-1}(tY))_{AB} c_B^i, \\ \text{so} \quad \left. \frac{d}{dt} \right|_{t=0} T_{gh(t)} &= \left. \frac{d}{dt} \right|_{t=0} T_g T_{h(t)}. \end{aligned}$$

By integrating the last equation one arrives at the desired representation property,  $T_{gh} = T_g T_h$ . (q.e.d.)

The subgroup  $H = \text{Gl}(n_+|n_+) \times \text{Gl}(n_-|n_-) \subset G$  consisting of elements  $h = \text{diag}(A, D)$  acts by

$$T_{\text{diag}(A,D)} = \exp (\bar{c}_{+a}^i (\ln A)_{ab} c_{+b}^i + \bar{c}_{-a}^i (\ln D)_{ab} c_{-b}^i).$$

This subgroup stabilizes the vacuum:

$$T_{\text{diag}(A,D)}|0\rangle = |0\rangle \exp N \sum_a (-1)^{|a|+1} (\ln D)_{aa} = |0\rangle \text{SDet} D^{-N}.$$

With  $\mu(\text{diag}(A,D)) := \text{SDet} D^{-N}$ , we say that the vacuum carries a one-dimensional representation  $\mu$  of  $H$ :

$$T_h|0\rangle = |0\rangle \mu(h), \quad \text{and} \quad \langle 0|T_h^{-1} = \mu(h)^{-1} \langle 0| \quad (h \in H).$$

After all these preliminaries we consider the generalized coherent states  $T_g|0\rangle$  ( $g \in G$ ) and, in particular, the operator  $P$  defined by the integral

$$P = \int_{M_B \times M_F} Dg_H T_g|0\rangle \langle 0|T_g^{-1}.$$

(Recall that  $Dg_H$  denotes the  $G$ -invariant Berezin measure of the coset space  $G/H$ , and the integration domain  $M_B \times M_F$  was specified in Sec. II B.) The integrand is a function on the coset space:

$$T_{gh}|0\rangle \langle 0|T_{gh}^{-1} = T_g T_h|0\rangle \langle 0|T_h^{-1} T_g^{-1} = T_g|0\rangle \langle 0|T_g^{-1} \quad (h \in H),$$

so the integral is well-defined. The following calculation shows that the operator  $P$  commutes with  $T_{g_0}$  for any  $g_0 \in G$ :

$$\begin{aligned} T_{g_0}P &= \int Dg_H T_{g_0} T_g|0\rangle \langle 0|T_g^{-1} \\ &= \int Dg_H T_{g_0 g}|0\rangle \langle 0|T_g^{-1} \\ &= \int Dg_H T_g|0\rangle \langle 0|T_{g_0^{-1}g}^{-1} \\ &= \int Dg_H T_g|0\rangle \langle 0|T_g^{-1} T_{g_0} = PT_{g_0}, \end{aligned}$$

where the invariance of  $Dg_H$  under translations  $g \mapsto g_0 g$  was used. (This invariance is crucial and is what determines  $D(Z, \tilde{Z}) = Dg_H$  in (3), up to multiplication by a constant.) As was explained earlier, the action of the Lie superalgebra of  $G$  on the flavor sector is irreducible. By Schur's lemma then, the fact that  $P$  commutes with all operators  $T_{g_0}$  leads to the conclusion that  $P$  is proportional to the identity on the flavor sector.

Let us determine the constant of proportionality. To that end, let  $\pi : G \rightarrow G/H$  be the canonical projection assigning group elements  $g$  to cosets  $\pi(g) \equiv gH$ . We make a decomposition  $g = s(\pi(g))h(g)$ , where  $s(\pi(g))$  takes values in  $G$  and  $h(g)$  takes values in  $H$ . (Such a decomposition fixes a ‘‘choice of gauge’’ [21,16], and  $s : G/H \rightarrow G$  is called a (local) section of the bundle  $\pi : G \rightarrow G/H$ .) In this way we get

$$\begin{aligned} T_g|0\rangle &= T_{s(\pi(g))} T_{h(g)}|0\rangle = T_{s(\pi(g))}|0\rangle \mu(h(g)), \\ \langle 0|T_g^{-1} &= \mu(h(g))^{-1} \langle 0|T_{s(\pi(g))}^{-1}, \quad \text{and} \quad T_g|0\rangle \langle 0|T_g^{-1} = T_{s(\pi(g))}|0\rangle \langle 0|T_{s(\pi(g))}^{-1}. \end{aligned}$$

Following the main text we put  $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$  and parameterize cosets  $\pi(g)$  by the pair  $(Z = BD^{-1}, \tilde{Z} = CA^{-1})$ . For  $s$  we choose

$$\begin{aligned} s(\pi(g)) \equiv s(Z, \tilde{Z}) &= \begin{pmatrix} (1 - Z\tilde{Z})^{-1/2} & Z(1 - \tilde{Z}Z)^{-1/2} \\ \tilde{Z}(1 - Z\tilde{Z})^{-1/2} & (1 - \tilde{Z}Z)^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (1 - Z\tilde{Z})^{+1/2} & 0 \\ 0 & (1 - \tilde{Z}Z)^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tilde{Z} & 1 \end{pmatrix}. \end{aligned}$$

The first line shows that this is a valid choice of section, i.e. we indeed have  $Z = BD^{-1}$  and  $\tilde{Z} = CA^{-1}$ . Using the second line to translate  $s(Z, \tilde{Z})$  into an operator on Fock space, we obtain

$$\begin{aligned} T_{s(Z, \tilde{Z})}|0\rangle &= \exp(\bar{c}_{+a}^i Z_{ab} c_{-b}^i) \exp\left(\frac{1}{2} \bar{c}_{+a}^i \ln(1 - Z\tilde{Z})_{ab} c_{+b}^i\right. \\ &\quad \left. - \frac{1}{2} \bar{c}_{-a}^i \ln(1 - \tilde{Z}Z)_{ab} c_{-b}^i\right) \exp(\bar{c}_{-a}^i \tilde{Z}_{ab} c_{+b}^i) |0\rangle \\ &= \exp(\bar{c}_{+a}^i Z_{ab} c_{-b}^i) |0\rangle \text{SDet}(1 - \tilde{Z}Z)^{N/2} =: |Z\rangle, \quad \text{and} \\ \langle 0|T_{s(Z, \tilde{Z})}^{-1} &= \langle 0|T_{s(-Z, -\tilde{Z})} = \text{SDet}(1 - \tilde{Z}Z)^{N/2} \langle 0| \exp(-\bar{c}_{-a}^i \tilde{Z}_{ab} c_{+b}^i) =: \langle Z|. \end{aligned}$$

The expression for the invariant measure  $Dg_H$  in the coordinates  $Z, \tilde{Z}$  is  $D(Z, \tilde{Z})$ , so

$$P = \int_{M_B \times M_F} Dg_H T_g |0\rangle \langle 0| T_g^{-1} = \int D(Z, \tilde{Z}) T_{s(Z, \tilde{Z})} |0\rangle \langle 0| T_{s(Z, \tilde{Z})}^{-1} = \int D(Z, \tilde{Z}) |Z\rangle \langle Z|.$$

Taking the vacuum expectation value of  $P$  we find

$$\langle 0|P|0\rangle = \int D(Z, \tilde{Z}) \langle 0|Z\rangle \langle Z|0\rangle = \int D(Z, \tilde{Z}) \text{SDet}(1 - \tilde{Z}Z)^N = \int D\mu_N(Z, \tilde{Z}) = 1,$$

by our choice of normalization, see Sec. II A. It follows that the constant of proportionality between  $P$  and the unit operator on the flavor sector is unity. Because  $\langle Z|$  vanishes on states that are not color-singlets,  $P$  annihilates all states outside the flavor sector. Hence  $P$  projects Fock space onto the flavor sector.

The final ingredient we need are the Bose-Fermi coherent states

$$\exp(\bar{c}_{+a}^i \psi_{+a}^i + \bar{\psi}_{-b}^i c_{-b}^i) |0\rangle.$$

Note that the ordering in the exponential matters as the Grassmann variables are taken to anticommute with the fermionic operators:  $\bar{c}_{+a}^i \psi_{+a}^i = (-1)^{|a|} \psi_{+a}^i \bar{c}_{+a}^i$ , as before. These Bose-Fermi coherent states span the entire Fock space (or, rather, the “Grassmann envelope” [38] thereof). They can be projected on the flavor (or color-neutral) sector by making a unitary rotation in color space,  $c_A^i \mapsto c_A^j U^{ji}$ ,  $\bar{c}_A^i \mapsto \bar{U}^{ji} \bar{c}_A^j$ , and averaging over all such rotations. Therefore the operator  $P$  that projects on the flavor sector acts on Bose-Fermi coherent states as

$$P \exp(\bar{c}_{+a}^i \psi_{+a}^i + \bar{\psi}_{-b}^i c_{-b}^i) |0\rangle = \int_{U(N)} dU \exp(\bar{U}^{ji} \bar{c}_{+a}^j \psi_{+a}^i + \bar{\psi}_{-b}^i c_{-b}^j U^{ji}) |0\rangle.$$

With all these tools in hand, the statement (3) is proved by the following computation:

$$\begin{aligned} & \int D\mu_N(Z, \tilde{Z}) \exp(\bar{\psi}_{+a}^i Z_{ab} \psi_{-b}^i + \bar{\psi}_{-b}^i \tilde{Z}_{ba} \psi_{+a}^i) \\ &= \int D\mu_N(Z, \tilde{Z}) \langle 0| \exp(\bar{\psi}_{+a}^i c_{+a}^i - \bar{c}_{-b}^i \psi_{-b}^i) \exp(\bar{c}_{+a}^i Z_{ab} c_{-b}^i) |0\rangle \\ & \quad \times \langle 0| \exp(-\bar{c}_{-b}^i \tilde{Z}_{ba} c_{+a}^i) \exp(\bar{c}_{+a}^i \psi_{+a}^i + \bar{\psi}_{-b}^i c_{-b}^i) |0\rangle \\ &= \langle 0| \exp(\bar{\psi}_{+a}^i c_{+a}^i - \bar{c}_{-b}^i \psi_{-b}^i) \left( \int D(Z, \tilde{Z}) |Z\rangle \langle Z| \right) \exp(\bar{c}_{+a}^i \psi_{+a}^i + \bar{\psi}_{-b}^i c_{-b}^i) |0\rangle \\ &= \langle 0| \exp(\bar{\psi}_{+a}^i c_{+a}^i - \bar{c}_{-b}^i \psi_{-b}^i) P \exp(\bar{c}_{+a}^i \psi_{+a}^i + \bar{\psi}_{-b}^i c_{-b}^i) |0\rangle \\ &= \int_{U(N)} dU \langle 0| \exp(\bar{\psi}_{+a}^i c_{+a}^i - \bar{c}_{-b}^i \psi_{-b}^i) \exp(\bar{c}_{+a}^i U^{ij} \psi_{+a}^j + \bar{\psi}_{-b}^i \bar{U}^{ji} c_{-b}^j) |0\rangle \\ &= \int_{U(N)} dU \exp(\bar{\psi}_{+a}^i U^{ij} \psi_{+a}^j + \bar{\psi}_{-b}^i \bar{U}^{ji} \psi_{-b}^j). \end{aligned}$$

The first equality sign is an elementary consequence of the commutation relations of the Fock operators  $c, \bar{c}$  and their action on the vacuum  $|0\rangle$ . The second and third equality signs recognize the projector  $P$  on the flavor sector. The fourth equality sign implements  $P$  by averaging over all unitary rotations in color space. The last equality sign is elementary again.

- [1] F. Wegner, Z. Phys. **B35**, 207 (1979).
- [2] K.B. Efetov, Adv. Phys. **32**, 53 (1983).
- [3] B.A. Muzykantskii and D.E. Khmelnitskii, Phys. Rev. **B51**, 5480 (1995).
- [4] A. V. Andreev, O. Agam, B. D. Simons and B. L. Altshuler, cond-mat/9605204.
- [5] D.K.K. Lee and J.T. Chalker, Phys. Rev. Lett. **72**, 1510 (1994).
- [6] F.M. Izrailev, Phys. Rep. **196**, 299 (1990).

- [7] F. Haake, *Quantum signatures of chaos*, Springer-Verlag, Berlin 1991.
- [8] A. Altland and M.R. Zirnbauer, submitted to Phys. Rev. Lett. (June 1996).
- [9] M.R. Zirnbauer, unpublished.
- [10] F.J. Dyson, J. Math. Phys. **3**, 140 (1962).
- [11] M.J. Rothstein, Trans. Am. Math. Soc. **299**, 387 (1987).
- [12] M.R. Zirnbauer, to appear in J. Math. Phys. (October 1996).
- [13] F.A. Berezin and D.A. Leites, Sov. Math. Dokl. **16**, 1218 (1975).
- [14] B. Kostant, Lect. Notes Math. **570**, 177 (1977).
- [15] H.A. Weidenmüller and M.R. Zirnbauer, Nucl. Phys. B**305**, 339 (1988).
- [16] M.R. Zirnbauer, Ann. d. Physik **3**, 513 (1994); cond-mat/9410040.
- [17] M.L. Mehta, *Random Matrices*, Academic Press, New York 1991.
- [18] J.J.M. Verbaarschot, H.A. Weidenmüller, and M.R. Zirnbauer, Phys. Rep. **129**, 367 (1985).
- [19] C. Itzykson and J.B. Zuber, J. Math. Phys. **21**, 411 (1980).
- [20] Harish-Chandra, Am. J. Math. **79**, 87 (1957).
- [21] M. Stone, Nucl. Phys. B**314**, 557 (1989).
- [22] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York 1978.
- [23] D. Fuchs, Ph. D. thesis, Universität zu Köln, 1994.
- [24] T. Guhr, Commun. Math. Phys. **176**, 555 (1996).
- [25] J. Alfaro, R. Medina, and L.F. Urrutia, J. Math. Phys. **36**, 3085 (1995).
- [26] T. Guhr, J. Math. Phys. **32**, 336 (1991).
- [27] M.R. Zirnbauer and F.D.M. Haldane, Phys. Rev. B**52**, 8729 (1995).
- [28] P.W. Brouwer and K. Frahm, Phys. Rev. B**53**, 1490 (1996).
- [29] R. Bundschuh, diploma thesis, Universität zu Köln, 1993.
- [30] F.J. Dyson, Commun. Math. Phys. **19**, 235 (1970).
- [31] A. Altland and M.R. Zirnbauer, cond-mat/9602137.
- [32] A. Altland and M.R. Zirnbauer, Phys. Rev. Lett. **76**, 3420 (1996).
- [33] O. Agam, A. Altland, and M.R. Zirnbauer, unpublished.
- [34] G. 't Hooft, Nucl. Phys. B**72**, 461 (1974); Nucl. Phys. B**75**, 461 (1975); E. Witten, Nucl. Phys. B**160**, 57 (1979).
- [35] A.M. Perelomov, Commun. Math. Phys. **26**, 222 (1972).
- [36] E. Onofri, J. Math. Phys. **16**, 1087 (1975).
- [37] A.M. Perelomov, *Generalized coherent states and their applications*, Springer-Verlag, Berlin 1986.
- [38] F.A. Berezin, *Introduction to Superanalysis*, Reidel, Dordrecht 1987.